Arbitrage-free SVI volatility surfaces

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(Including joint work with Antoine Jacquier)

Outline

- No-arbitrage constraints on the tail behavior of implied volatility
- History of SVI
- Equivalent representations
- How to eliminate calendar spread arbitrage
- Butterfly arbitrage
- Simple closed-form arbitrage-free SVI surfaces
- How to eliminate butterfly arbitrage
- How to interpolate and extrapolate
- Calibration of SPX volatility surface
- An alternative to SABR?

Roger Lee's Moment Formula

- [11] shows that implied variance is bounded above by a function linear in the log-strike $k = \log(K/F)$ as $|k| \to \infty$.
  - The maximum slope of total implied variance $w(k, T) = \sigma_{BS}^2(k, T)\, T$ is 2.
- He shows how to relate the gradients of the wings of the upper bound of the implied variance skew to the maximal finite moments of the underlying process.
- Lee's derivation assumes only the existence of a martingale measure: it makes no assumptions on the distribution of underlying returns. His result is completely model-independent.

Roger Lee's Lemma 3.1

Lemma 1

There exists $k^\star > 0$ *such that for all* $k > k^\star$,

$$\sigma_{BS}^2(k) < 2\,\frac{|k|}{T}.$$

Roger Lee's Lemma 3.1

Proof.

We only need to show that
$C_{BS}\left(k,\sigma_{BS}(k)\sqrt{T}\right) < C_{BS}\left(k,\sqrt{2\,|k|}\right)$ whenever $k > k^\star$.

On the LHS, we have

$$\lim_{k\to\infty} C_{BS}\left(k,\sigma_{BS}(k)\sqrt{T}\right) = 0$$

and on the RHS, we have

$$\begin{aligned}\lim_{k\to\infty} C_{BS}\left(k,\sqrt{2\,|k|}\right) &= \lim_{k\to\infty} F\left\{N(d_1) - e^k\,N(d_2)\right\}\\ &= \lim_{k\to\infty} F\left\{N(0) - e^k\,N(-\sqrt{2\,|k|})\right\} = \frac{F}{2}\end{aligned}$$

□

Slope of left wing

Let $q^* := \sup \left\{ q : \mathbb{E}\, S_T^{-q} < \infty \right\}$ and

$$\beta^* := \limsup_{k \to -\infty} \frac{\sigma_{BS}^2(k, T)\, T}{|k|}$$

Then $\beta^* \in [0, 2]$,

$$q^* = \frac{1}{2} \left(\frac{1}{\sqrt{\beta^*}} - \frac{\sqrt{\beta^*}}{2} \right)^2$$

and inverting this, we obtain $\beta^* = g(q^*)$ with

$$g(x) = 2 - 4 \left[\sqrt{x^2 + x} - x \right]$$

Slope of right wing

Similarly, let $p^* := \sup \left\{ p : \mathbb{E}\, S_T^{1+p} < \infty \right\}$ and

$$\alpha^* := \limsup_{k \to +\infty} \frac{\sigma_{BS}^2(k, T)\, T}{|k|}$$

Then $\alpha^* \in [0, 2]$,

$$p^* = \frac{1}{2} \left(\frac{1}{\sqrt{\alpha^*}} - \frac{\sqrt{\alpha^*}}{2} \right)^2$$

and as for the left wing, it follows that $\alpha^* = g(p^*)$.

Implications of the moment formula

- Implied variance is linear in k as $k \to \infty$ for stochastic volatility models.
- So, if we want a parameterization of the implied variance surface consistent with stochastic volatility, it needs to be linear in the wings!
    - and it needs to be curved in the middle - many conventional parameterizations of the volatility surface are quadratic for example.

Benaim and Friz

- With $F(x)$ the CDF, Benaim and Friz go on to show that Roger Lee's upper bound (lim sup) may be replaced by a limit in most practical cases.
- Then we may write for the right tail

$$\frac{\sigma_{BS}(k,T)^2\,T}{k} \sim g\left(-1 - \frac{\log\left[1-F(k)\right]}{k}\right) \text{ as } k \to \infty \qquad (1)$$

and for the left tail

$$\frac{\sigma_{BS}(-k,T)^2\,T}{k} \sim g\left(\frac{-\log F(-k)}{k}\right) \text{ as } k \to \infty \qquad (2)$$

- By substituting the tail-behavior of F into equations (1) and (2), we can deduce the full tail behavior of the smile, not just Roger Lee's upper bound.

History of SVI

- SVI was originally devised at Merrill Lynch in 1999 and subsequently publicly disseminated in [4].
- SVI has two key properties that have led to its subsequent popularity with practitioners:
  - For a fixed time to expiry t, the implied Black-Scholes variance $\sigma^2_{\mathrm{BS}}(k,t)$ is linear in the log-strike k as $|k| \to \infty$ consistent with Roger Lee's moment formula [11].
  - It is relatively easy to fit listed option prices whilst ensuring no calendar spread arbitrage.
- The consistency of the SVI parameterization with arbitrage bounds for extreme strikes has also led to its use as an extrapolation formula [9].
- As shown in [6], the SVI parameterization is not arbitrary in the sense that the large-maturity limit of the Heston implied volatility smile is exactly SVI.

Previous work

- Calibration of SVI to given implied volatility data (for example [12]).
- [2] showed how to parameterize the volatility surface so as to preclude dynamic arbitrage.
- Arbitrage-free interpolation of implied volatilities by [1], [3], [8], [10].
- Prior work has not successfully attempted to eliminate static arbitrage.

- Efforts to find simple closed-form arbitrage-free parameterizations of the implied volatility surface are widely considered to be futile.

Notation

- Given a stock price process $(S_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$, the forward price process $(F_t)_{t\geq 0}$ is $F_t := \mathbb{E}\left(S_t|\mathcal{F}_0\right)$.
- For any $k \in \mathbb{R}$ and $t > 0$, $C_{\mathrm{BS}}(k, \sigma^2 t)$ denotes the Black-Scholes price of a European Call option on S with strike $F_t \mathrm{e}^k$, maturity t and volatility $\sigma > 0$.
- $\sigma_{\mathrm{BS}}(k, t)$ denotes Black-Scholes implied volatility.
- Total implied variance is $w(k, t) = \sigma_{\mathrm{BS}}^2(k, t)t$.
- The implied variance $v(k, t) = \sigma_{\mathrm{BS}}^2(k, t) = w(k, t)/t$.
- The map $(k, t) \mapsto w(k, t)$ is the volatility surface.
- For any fixed expiry $t > 0$, the function $k \mapsto w(k, t)$ represents a slice.

The raw SVI parameterization

For a given parameter set $\chi_R = \{a, b, \rho, m, \sigma\}$, the *raw SVI parameterization* of total implied variance reads:

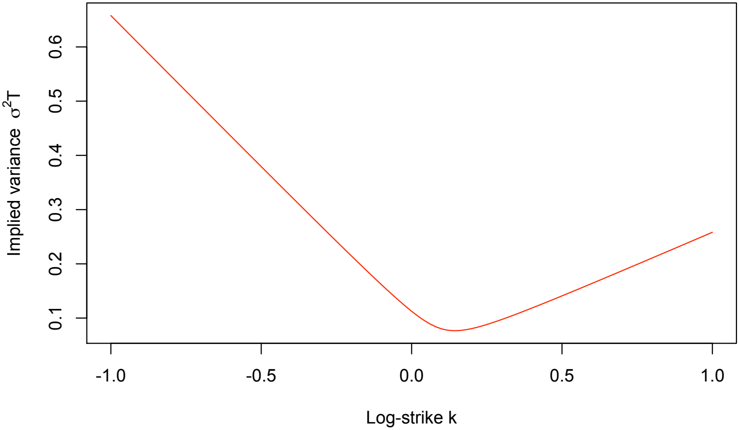
Raw SVI parameterization

$$w(k;\chi_R) = a + b\left\{\rho\,(k-m) + \sqrt{(k-m)^2+\sigma^2}\right\}$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$, $\sigma > 0$, and the obvious condition $a + b\,\sigma\,\sqrt{1-\rho^2} \geq 0$, which ensures that $w(k,\chi_R) \geq 0$ for all $k \in \mathbb{R}$. This condition ensures that the minimum of the function $w(\cdot,\chi_R)$ is non-negative.

An SVI example

Figure 1: With $a = 0.04$, $b = 0.4$, $\sigma = 0.1$, $\rho = -0.4$, $m = 0$, we obtain



Meaning of raw SVI parameters

Changes in the parameters have the following effects:

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

The natural SVI parameterization

For a given parameter set $\chi_N = \{\Delta, \mu, \rho, \omega, \zeta\}$, the *natural SVI parameterization* of total implied variance reads:

Natural SVI parameterization

$$w(k;\chi_N) = \Delta + \frac{\omega}{2}\left\{1 + \zeta\rho\,(k-\mu) + \sqrt{(\zeta(k-\mu)+\rho)^2 + (1-\rho^2)}\right\}$$

where $\omega \geq 0$, $\Delta \in \mathbb{R}$, $\mu \in \mathbb{R}$, $|\rho| < 1$ and $\zeta > 0$.

- This parameterization is a natural generalization of the time ∞ Heston smile explored in [6].

The SVI Jump-Wings (SVI-JW) parameterization

- Neither the raw SVI nor the natural SVI parameterizations are intuitive to traders.
- There is no reason to expect these parameters to be particularly stable.
- The *SVI-Jump-Wings (SVI-JW) parameterization* of the implied variance v (rather than the implied total variance w) was inspired by a similar parameterization attributed to Tim Klassen, then at Goldman Sachs.

SVI-JW

For a given time to expiry $t > 0$ and a parameter set $\chi_J = \{v_t, \psi_t, p_t, c_t, \widetilde{v}_t\}$ the SVI-JW parameters are defined from the raw SVI parameters as follows:

SVI-JW parameterization

$$
\begin{aligned}
v_t &= \frac{a + b\left\{-\rho\, m + \sqrt{m^2 + \sigma^2}\right\}}{t}, \\
\psi_t &= \frac{1}{\sqrt{w_t}} \frac{b}{2} \left(-\frac{m}{\sqrt{m^2 + \sigma^2}} + \rho\right), \\
p_t &= \frac{1}{\sqrt{w_t}} b\,(1 - \rho), \\
c_t &= \frac{1}{\sqrt{w_t}} b\,(1 + \rho), \\
\widetilde{v}_t &= \left(a + b\,\sigma\,\sqrt{1 - \rho^2}\right)/t
\end{aligned}
$$

with $w_t := v_t\, t$.

Interpretation of SVI-JW parameters

The SVI-JW parameters have the following interpretations:

- v_t gives the ATM variance;
- ψ_t gives the ATM skew;
- p_t gives the slope of the left (put) wing;
- c_t gives the slope of the right (call) wing;
- $\widetilde{v}_t$ is the minimum implied variance.

Scaling of SVI Jump-Wings parameters with volatility

Note that, as defined here,

$$\psi_t = \left.\frac{\partial \sigma_{BS}(k)}{\partial k}\right|_{k=0}$$

The choice of volatility skew as the skew measure rather than variance skew for example, reflects the empirical observation that volatility is roughly lognormally distributed. Specifically, we show in Chapter 7 of *The Volatility Surface* that if the SDE for variance is of the form:

$$dv = \alpha(v)\, dt + \eta \sqrt{v}\, \beta(v)\, dZ$$

we should have

$$\frac{\partial}{\partial k} \sigma_{BS}(k,T)^2 \approx \frac{\rho\, \eta\, \beta(v)}{\lambda'\, T} \left\{ 1 - \frac{\left(1 - e^{-\lambda' T}\right)}{\lambda'\, T} \right\} \propto \beta(v)$$

with $\lambda' = \lambda - \frac{1}{2}\, \rho\, \eta\, \beta(v)$.

Scaling of SVI Jump-Wings parameters with volatility

Thus

$$\left.\frac{\partial \sigma_{BS}(k)}{\partial k}\right|_{k=0} \approx \text{const.}$$

independent of volatility implies that

$$\beta(v) \sim \sqrt{v}$$

and therefore that the variance (volatility) process is lognormal.

This consistency of the SVI-JW parameterization with empirical volatility dynamics leads to greater parameter stability over time.

Scaling of SVI Jump-Wings parameters with time to expiration

- If smiles scaled perfectly as $1/\sqrt{w_t}$ (effectively $1/\sqrt{t}$ in practice), SVI-JW parameters would be constant, independent of the slice t.
  - This makes it easy to extrapolate the SVI surface to expirations beyond the longest expiration in the data set.

- Since both scaling features are roughly consistent with empirical observation, we expect (and see) greater parameter stability over time.
  - Traders can keep parameters in their heads.

Inversion of SVI Jump-Wings parameters

$$\begin{aligned} b &= \frac{\sqrt{w_t}}{2}\,(c_t + p_t) \\ \rho &= 1 - \frac{p_t\,\sqrt{w_t}}{b} \end{aligned}$$

Define $\alpha := \sigma/m$. Then

$$\beta := \rho - \frac{\psi_t\,\sqrt{w_t}}{b} = \frac{\operatorname{sign}(\alpha)}{\sqrt{1+\alpha^2}}$$

Solving this equation gives

$$\alpha = \operatorname{sign}(\beta)\sqrt{\frac{1}{\beta^2} - 1}$$

SVI-JW inversion continued

We now note that

$$\frac{(v_t - \tilde{v}_t)\ t}{b} = m\left\{-\rho + \operatorname{sign}(\alpha)\sqrt{1+\alpha^2} - \alpha\sqrt{1-\rho^2}\right\}$$

from which we can deduce m. Finally

$$\begin{aligned} \sigma &= \alpha\, m \\ a &= \tilde{v}_t\, t - b\,\sigma\sqrt{1-\rho^2} \end{aligned}$$

- Any one of the three versions of the SVI parameterization can be easily transformed into any of the others.

Elimination of calendar spread arbitrage

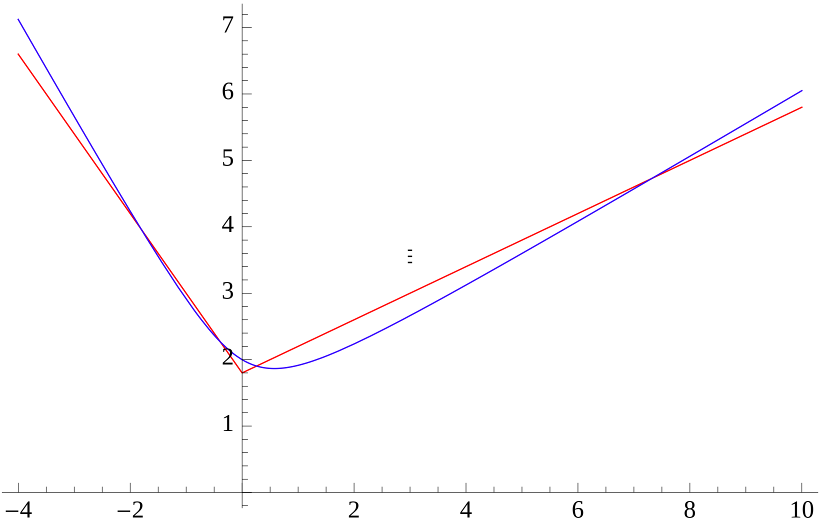
Lemma 2

If dividends are proportional to the stock price, the volatility surface w is free of calendar spread arbitrage if and only if

$$\partial_t w(k,t) \geq 0, \quad \text{for all } k \in \mathbb{R} \text{ and } t > 0.$$

- Thus there is no calendar spread arbitrage if there are no crossed lines on a total variance plot.

SVI slices may cross at no more than four points



Condition for no calendar spread arbitrage

Lemma 3

Two raw SVI slices admit no calendar spread arbitrage if a certain quartic polynomial has no real root.

Ferrari Cardano

The idea is as follows:

- Two total variance slices cross if

$$
\begin{aligned}
& a_1 + b_1 \left\{ \rho_1 \left(k - m_1\right) + \sqrt{\left(k - m_1\right)^2 + \sigma_1^2} \right\} \\
= \quad & a_2 + b_2 \left\{ \rho_2 \left(k - m_2\right) + \sqrt{\left(k - m_2\right)^2 + \sigma_2^2} \right\}
\end{aligned}
$$

- Rearranging and squaring gives a quartic polynomial equation of the form

$$
\alpha_4\, k^4 + \alpha_3\, k^3 + \alpha_2\, k^2 + \alpha_1\, k + \alpha_0 = 0,
$$

  where each of the coefficients are lengthy yet explicit expressions in terms of the raw SVI parameters.

  - If this quartic polynomial has no real root, then the slices do not intersect.

Butterfly arbitrage

Definition 4

A slice is said to be free of butterfly arbitrage if the corresponding density is non-negative.

Now introduce the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}.$$

Lemma 5

A slice is free of butterfly arbitrage if and only if $g(k) \geq 0$ for all $k \in \mathbb{R}$ and $\lim_{k \to +\infty} d_+(k) = -\infty$.

Axel Vogt post on Wilmott.com

AVt
Senior Member

Posts: 971
Joined: Dec 2001

Thu Apr 06, 06 08:37 PM

It works for observables and far beyond for extrapolation.

But for a (theoretical) experiment try the following data

a = -.40998372001772e-1,
b = .13308181151379,
m = .35858898335748,
rho = .30602086142471,
sigma = .41531878803777

The Vogt smile

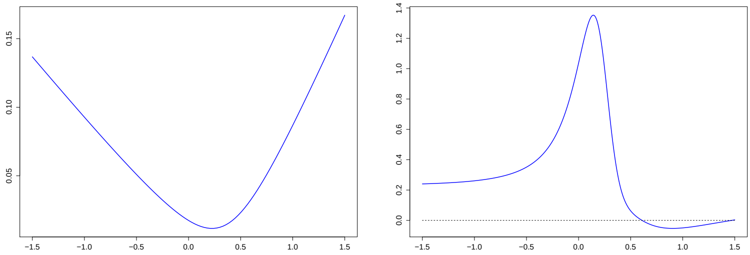


Figure 2: Plots of the total variance smile w (left) and the function g (right), using Axel Vogt's parameters

Simple SVI

Consider now the following extension of the natural SVI parameterization:

Simple SVI (SSVI) parameterization

$$w(k,\theta_t) = \frac{\theta_t}{2}\left\{1+\rho\varphi(\theta_t)k+\sqrt{(\varphi(\theta_t)k+\rho)^2+(1-\rho^2)}\right\} \quad (3)$$

with $\theta_t > 0$ for $t > 0$, and where φ is a smooth function from $(0,\infty)$ to $(0,\infty)$ such that the limit $\lim_{t\to 0}\theta_t\varphi(\theta_t)$ exists in $\mathbb{R}$.

Interpretation of SSVI

- This representation amounts to considering the volatility surface in terms of ATM variance time, instead of standard calendar time.
- The ATM total variance is $\theta_t = \sigma_{\mathrm{BS}}^2(0,t)\,t$ and the ATM volatility skew is given by

$$\partial_k \sigma_{\mathrm{BS}}(k,t)|_{k=0} = \frac{1}{2\sqrt{\theta_t t}} \partial_k w(k,\theta_t)\Big|_{k=0} = \frac{\rho\sqrt{\theta_t}}{2\sqrt{t}}\varphi(\theta_t).$$

- The smile is symmetric around at-the-money if and only if $\rho = 0$, a well-known property of stochastic volatility models.

Conditions on SSVI for no calendar spread arbitrage

Theorem 6

The surface (3) *is free of calendar spread arbitrage if*

1. $\partial_t \theta_t \geq 0$, *for all* $t \geq 0$;
2. $\partial_\theta(\theta\varphi(\theta)) \geq 0$, *for all* $\theta > 0$;
3. $\partial_\theta \varphi(\theta) < 0$, *for all* $\theta > 0$.

- Simple SVI (3) is free of calendar spread arbitrage if:
  - the skew in total variance terms is monotonically increasing in trading time and
  - the skew in implied variance terms is monotonically decreasing in trading time.
- In practice, any reasonable skew term structure that a trader defines will have these properties.

Idea of proof

The proof proceeds by computing, for any $\theta > 0$,

$$2\partial_\theta w(k,\theta) = 1 + \frac{1+\rho x}{\sqrt{x^2+2\rho x+1}} + \frac{\theta\varphi'(\theta)+\varphi(\theta)}{\varphi(\theta)}\left\{\frac{x+\rho}{\sqrt{x^2+2\rho x+1}}+\rho\right\} \tag{4}$$

with $x := k\,\varphi(\theta)$ and noting that

$$0 \leq \frac{\theta\varphi'(\theta)+\varphi(\theta)}{\varphi(\theta)} \leq 1.$$

Are the conditions necessary?

- The necessity of Condition 1 follows from imposing $\partial_t w(0;\theta_t) \geq 0$.
- $\partial_\theta w(k,\theta) \geq 0$ (with $x = k\varphi(\theta)$) imposes the necessity of condition 2.
- That condition 3 is not necessary can be seen by setting $\rho = 0$ in (4) to give

$$2\partial_\theta w(k,\theta) = 1 + \frac{1}{\sqrt{1+x^2}} + \frac{\theta\varphi'(\theta)+\varphi(\theta)}{\varphi(\theta)}\,\frac{x^2}{\sqrt{1+x^2}},$$

which is positive if condition 2 holds whether or not condition 3 also holds.

Conditions on SSVI for no butterfly arbitrage

Theorem 7

The volatility surface (3) is free of butterfly arbitrage if the following conditions are satisfied for all $\theta > 0$*:*

1. $\theta\varphi(\theta)\left(1 + |\rho|\right) < 4$;
2. $\theta\varphi(\theta)^2\left(1 + |\rho|\right) \leq 4$.

Remark 8

Condition 1 needs to be a strict inequality so that $\lim_{k \to +\infty} d_+(k) = -\infty$ and the SVI density integrates to one.

Are these conditions necessary?

Lemma 9

The volatility surface (3) is free of butterfly arbitrage only if

$$\theta\varphi(\theta)\,(1+|\rho|) \leq 4, \quad \textit{for all } \theta > 0.$$

Moreover, if $\theta\varphi(\theta)\,(1+|\rho|) = 4$*, the surface (3) is free of butterfly arbitrage only if*

$$\theta\varphi(\theta)^2\,(1+|\rho|) \leq 4.$$

So the theorem is almost if-and-only-if.

No butterfly arbitrage in terms of SVI-JW parameters

A volatility smile of the form (3) is free of butterfly arbitrage if

$$\sqrt{v_t\, t}\max\left(p_t, c_t\right) < 4, \quad \text{and} \quad \left(p_t + c_t\right)\max\left(p_t, c_t\right) \leq 8,$$

hold for all $t > 0$.

The Roger Lee arbitrage bounds

- The asymptotic behavior of the surface (3) as $|k|$ tends to infinity is

$$w(k, \theta_t) = \frac{(1 \pm \rho)\, \theta_t}{2} \varphi(\theta_t)\, |k| + \mathcal{O}(1), \quad \text{for any } t > 0.$$

- Thus the condition $\theta\varphi(\theta)\,(1 + |\rho|) \leq 4$ of Theorem 7 corresponds to the upper bound of 2 on the asymptotic slope established by Lee [11].
  - Again, Condition 1 of the theorem is necessary.

No static arbitrage with SSVI

Corollary 5.1

The surface (3) is free of static arbitrage if the following conditions are satisfied:

1. $\partial_t \theta_t \geq 0$, for all $t > 0$
2. $\partial_\theta (\theta \varphi(\theta)) \geq 0$, for all $\theta > 0$;
3. $\partial_\theta \varphi(\theta) < 0$, for all $\theta > 0$;
4. $\theta \varphi(\theta) (1 + |\rho|) < 4$, for all $\theta > 0$;
5. $\theta \varphi(\theta)^2 (1 + |\rho|) \leq 4$, for all $\theta > 0$.

- A large class of simple closed-form arbitrage-free volatility surfaces!

A Heston-like surface

Example 10

The function φ defined as

$$\varphi(\theta) = \frac{1}{\lambda\theta}\left\{1 - \frac{1 - e^{-\lambda\theta}}{\lambda\theta}\right\},$$

with $\lambda \geq (1 + |\rho|)/4$ satisfies the conditions of Corollary 5.1.

- This function is consistent with the implied variance skew in the Heston model as shown in [5] (equation 3.19).

A power-law surface

Example 11

The choice

$$\varphi(\theta) = \frac{\eta}{\theta^{\gamma}\,(1+\theta)^{1-\gamma}}$$

gives a surface that is completely free of static arbitrage provided that $\gamma \in (0, 1/2]$ and $\eta\,(1+|\rho|) \leq 2$.

- This function is more consistent with the empirically-observed term structure of the volatility skew.

Even more flexibility...

Theorem 12

Let the volatility surface (3) satisfy the conditions of Corollary 5.1. If $\alpha_t \geq 0$ and $\partial_t \alpha_t \geq 0$, for all $t > 0$, then the volatility surface $w_\alpha(k, \theta_t) := w(k, \theta_t) + \alpha_t$ is free of static arbitrage.

- Corollary 5.1 gives us the freedom to match three features of one smile (level, skew, and curvature say) but only two features of all the other smiles (level and skew say), subject of course to the given smiles being themselves arbitrage-free.
- Theorem 12 may allow us to match an additional feature of each smile through α_t.

How to eliminate butterfly arbitrage

- We have shown how to define a volatility smile that is free of butterfly arbitrage.
- This smile is completely defined given three observables.
  - The ATM volatility and ATM skew are obvious choices for two of them.
  - The most obvious choice for the third observable in equity markets would be the asymptotic slope for k negative and in FX markets and interest rate markets, perhaps the ATM curvature of the smile might be more appropriate.

How to fix butterfly arbitrage

- Supposing we choose to fix the SVI-JW parameters v_t, ψ_t and p_t of a given SVI smile, we may guarantee a smile with no butterfly arbitrage by choosing the remaining parameters c'_t and $\widetilde{v}'_t$ according to SSVI as

$$c'_t = p_t + 2\,\psi_t, \quad \text{and} \quad \widetilde{v}'_t = v_t\,\frac{4p_t c'_t}{(p_t + c'_t)^2}.$$

- That is, given a smile defined in terms of its SVI-JW parameters, we are guaranteed to be able to eliminate butterfly arbitrage by changing the call wing c_t and the minimum variance $\widetilde{v}_t$, both parameters that are hard to calibrate with available quotes in equity options markets.

Example: Fixing the Vogt smile

- The SVI-JW parameters corresponding to the Vogt smile are:

$$\begin{aligned} & (v_t, \psi_t, p_t, c_t, \widetilde{v}_t) \\ = \; & (0.01742625, -0.1752111, 0.6997381, 1.316798, 0.0116249) \,. \end{aligned}$$

- We know then that choosing $(c_t, \widetilde{v}_t) = (0.3493158, 0.01548182)$ must give a smile free of butterfly arbitrage.
- There must exist some pair of parameters $\{c_t, \widetilde{v}_t\}$ with $c_t \in (0.349, 1.317)$ and $\widetilde{v}_t \in (0.0116, 0.0155)$ such that the new smile is free of butterfly arbitrage and is as close as possible to the original one in some sense.

Numerical optimization

- In this particular case, choosing the objective function as the sum of squared option price differences plus a large penalty for butterfly arbitrage, we arrive at the following "optimal" choices of the call wing and minimum variance parameters that still ensure no butterfly arbitrage:

$$(c_t, \widetilde{v}_t) = (0.8564763, 0.0116249) .$$

- Note that the optimizer has left $\widetilde{v}_t$ unchanged but has decreased the call wing.
- The resulting smiles and plots of the function g are shown in Figure 3.

The Vogt smile fixed

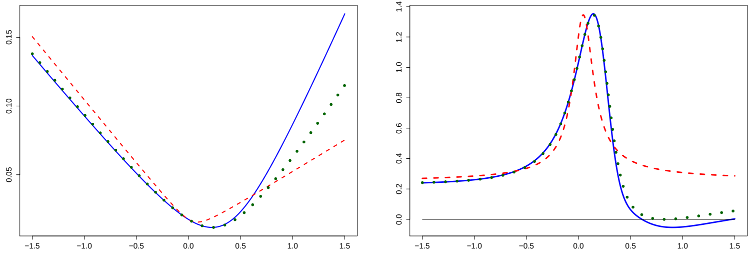


Figure 3: Plots of the total variance smile (left) and the function g (right). The graphs corresponding to the original Axel Vogt parameters is solid, to the guaranteed butterfly-arbitrage-free parameters dashed, and to the "optimal" choice of parameters dotted.

Why extra flexibility may not help

- The additional flexibility potentially afforded to us through the parameter α_t of Theorem 12 sadly does not help us with the Vogt smile.
- For α_t to help, we must have $\alpha_t > 0$; it is straightforward to verify that this translates to the condition $v_t\,(1-\rho^2) < \tilde{v}_t$ which is violated in the Vogt case.

Quantifying lines crossing

- Consider two SVI slices with parameters χ_1 and χ_2 where $t_2 > t_1$.
- We first compute the points k_i $(i = 1, \ldots, n)$ with $n \leq 4$ at which the slices cross, sorting them in increasing order. If $n > 0$, we define the points $\widetilde{k}_i$ as

$$\begin{aligned} \widetilde{k}_1 &:= k_1 - 1, \\ \widetilde{k}_i &:= \frac{1}{2}\left(k_{i-1} + k_i\right), \quad \text{if } 2 \leq i \leq n, \\ \widetilde{k}_{n+1} &:= k_n + 1. \end{aligned}$$

- For each of the $n + 1$ points $\widetilde{k}_i$, we compute the amounts c_i by which the slices cross:

$$c_i = \max\left[0, w(\widetilde{k}_i, \chi_1) - w(\widetilde{k}_i, \chi_2)\right].$$

Crossedness

Definition 13

The *crossedness* of two SVI slices is defined as the maximum of the c_i $(i = 1, \ldots, n)$. If $n = 0$, the crossedness is null.

A sample calibration recipe

Calibration recipe

- Given mid implied volatilities $\sigma_{ij} = \sigma_{\mathrm{BS}}(k_i, t_j)$, compute mid option prices using the Black-Scholes formula.
- Fit the square-root SVI surface by minimizing sum of squared distances between the fitted prices and the mid option prices. This is now the initial guess.
- Starting with the square-root SVI initial guess, change SVI parameters slice-by slice so as to minimize the sum of squared distances between the fitted prices and the mid option prices with a big penalty for crossing either the previous slice or the next slice (as quantified by the crossedness from Definition 13).

Interpolation

Lemma 14

Given two SVI smiles $w(k, t_1)$ and $w(k, t_2)$ with $t_1 < t_2$ where the two smiles are free of butterfly arbitrage and such that $w(k, \tau_2) \geq w(k, \tau_1)$ for all k, there exists an interpolation such that the interpolated volatility surface is free of static arbitrage for $t_1 < t < t_2$.

- For example;

$$\frac{C_t}{K_t} = \alpha_t \frac{C_1}{K_1} + (1 - \alpha_t) \frac{C_2}{K_2},$$

where for any $t \in (t_1, t_2)$, we define

$$\alpha_t := \frac{\sqrt{\theta_{t_2}} - \sqrt{\theta_t}}{\sqrt{\theta_{t_2}} - \sqrt{\theta_{t_1}}} \in [0, 1] \,.$$

works.

A possible choice of extrapolation

- At time $t_0 = 0$, the value of a call option is just the intrinsic value.
- Then we can interpolate between t_0 and t_1 using the above algorithm, guaranteeing no static arbitrage.
- For extrapolation beyond the final slice, first recalibrate the final slice using the simple SVI form (3).
- Then fix a monotonic increasing extrapolation of θ_t and extrapolate the smile for $t > t_n$ according to

$$w(k, \theta_t) = w(k, \theta_{t_n}) + \theta_t - \theta_{t_n},$$

which is free of static arbitrage if $w(k, \theta_{t_n})$ is free of butterfly arbitrage by Theorem 12.

Raw data

Raw option price data looks like this:

```
> spxData[100:105,]
    OPRA_Message_Sequence    Date     Time Exchange Message_Type Option_Root Expiration_Month_Code
100             157179374 9/15/11 150007.2        C           NA         SPX                     F
101             157180023 9/15/11 150007.2        C           NA         SPX                     F
102             157180136 9/15/11 150007.2        C           NA         SPX                     F
103             157180135 9/15/11 150007.2        C           NA         SPX                     F
104             157180220 9/15/11 150007.2        C           NA         SPX                     F
105             155910096 9/15/11 145524.8        C           NA         SPX                     I
    Expiration_Day Expiration_Year Strike_Price Option_Bid_Price Option_Bid_Size Option_Offer_Price
100             22              13          750            457.4              10              461.3
101             22              13          800            416.7              10              420.6
102             22              13          850            377.2              10              381.1
103             22              13          900            338.8              10              342.7
104             22              13          950            301.9              10              305.8
105             17              11          100           1105.9             100             1109.8
    Option_Offer_Size Session_Indicator Best_Bid.Offer_.BBO._Indicator BBO_Appendage_Exchange
100                10                NA                          FALSE                     NA
101                10                NA                          FALSE                     NA
102                10                NA                          FALSE                     NA
103                10                NA                          FALSE                     NA
104                10                NA                          FALSE                     NA
105               150                NA                          FALSE                     NA
    BBO_Appendage_Quote_Price BBO_Appendage_Quote_Size
100                        NA                       NA
101                        NA                       NA
102                        NA                       NA
103                        NA                       NA
104                        NA                       NA
105                        NA                       NA
```

Implied volatility computation for index options

- We compute all implied volatilities from option price data
  - We don't need external estimates of interest rates and dividends
- We use put-call parity to get implied forward prices and discount factors.
  - Find the unique forward price and discount factor that minimize implied forward pricing errors.
- In this way, we can avoid errors due to non-synchronous parameter estimates and typically generate very smooth implied volatility curves.

Implied volatility output

The resulting implied volatility output looks like this:

```
> spxOptData[90:110,]
        Expiry        Texp Strike       Bid       Ask      Fwd    CallMid
90  2011-09-16 0.002737851   1105 0.6428147 0.6944342 1207.695 102.769180
91  2011-09-16 0.002737851   1110 0.6133797 0.6630673 1207.695  97.769180
92  2011-09-16 0.002737851   1115 0.5839531 0.6316966 1207.695  92.769180
93  2011-09-16 0.002737851   1120 0.5545262 0.6317986 1207.695  87.794027
94  2011-09-16 0.002737851   1125 0.5689019 0.5990741 1207.695  82.818874
95  2011-09-16 0.002737851   1130 0.5374541 0.5662994 1207.695  77.818874
96  2011-09-16 0.002737851   1135 0.5059541 0.5553092 1207.695  72.843722
97  2011-09-16 0.002737851   1140 0.4743855 0.5213325 1207.695  67.843722
98  2011-09-16 0.002737851   1145 0.4427291 0.5039799 1207.695  62.868569
99  2011-09-16 0.002737851   1150 0.4109624 0.4529915 1207.695  57.843722
100 2011-09-16 0.002737851   1155 0.4010179 0.4334988 1207.695  52.893417
101 2011-09-16 0.002737851   1160 0.3839259 0.3979229 1207.695  47.918264
102 2011-09-16 0.002737851   1165 0.3620577 0.4025569 1207.695  43.042501
103 2011-09-16 0.002737851   1170 0.3258372 0.3924144 1207.695  38.141890
104 2011-09-16 0.002737851   1175 0.3237599 0.3443333 1207.695  33.216432
105 2011-09-16 0.002737851   1180 0.2900039 0.3134920 1207.695  28.290974
106 2011-09-16 0.002737851   1185 0.2537705 0.3047703 1207.695  23.514601
107 2011-09-16 0.002737851   1190 0.2556712 0.2698104 1207.695  18.887311
108 2011-09-16 0.002737851   1195 0.2228864 0.2674955 1207.695  14.483648
109 2011-09-16 0.002737851   1200 0.2210335 0.2593233 1207.695  10.651474
110 2011-09-16 0.002737851   1205 0.1989225 0.2471716 1207.695   7.067774
```

Total variance plot from fit to each slice independently

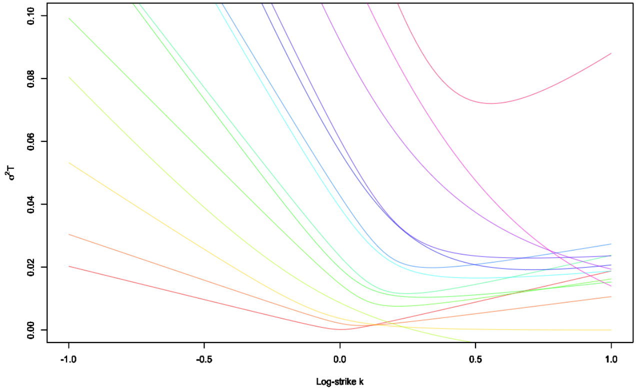


Figure 4: Fitting each slice independently gives rise to calendar spread arbitrage (crossed lines)

Square-root SVI calibration

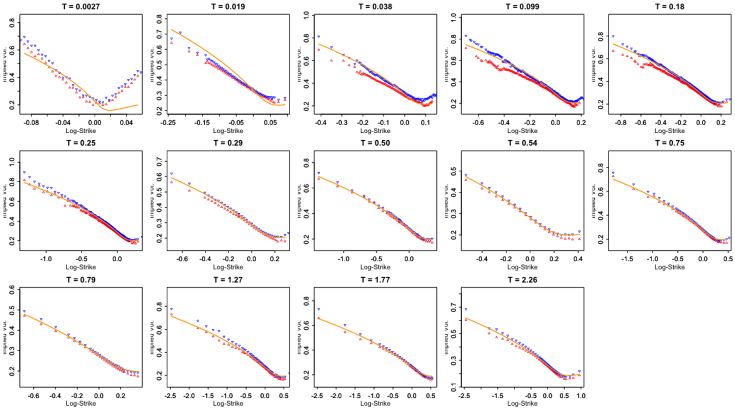


Figure 5: SPX option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the Sqrt SVI fit

SVI square-root calibration: December 2011 detail

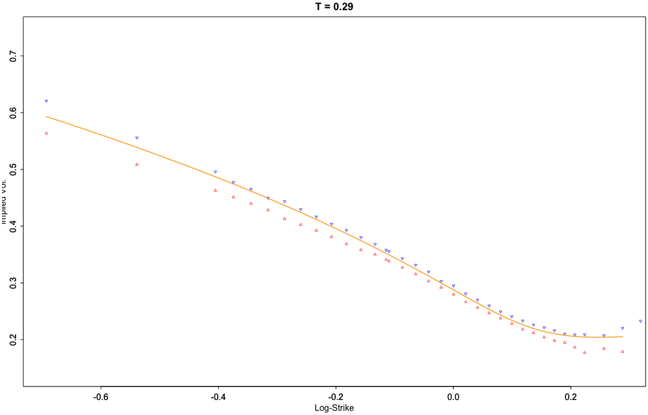


Figure 6: SPX Dec-2011 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the square-root SVI fit

SVI square-root calibration: Total variance plot

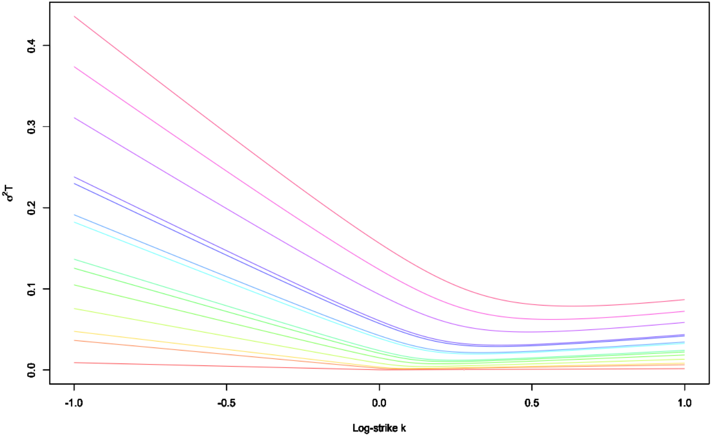


Figure 7: Total variance plot for square-root SVI fit: No lines cross!

JW parameters for square-root fit

	vt	psit	pt	ct	varmint	texp
1	0.05081151	-0.3100192	0.7499526	0.1299141	0.02557869	0.002737851
2	0.11101927	-0.3100192	0.7499526	0.1299141	0.05588749	0.019164956
3	0.09193989	-0.3100192	0.7499526	0.1299141	0.04628287	0.038329911
4	0.08456379	-0.3100192	0.7499526	0.1299141	0.04256971	0.098562628
5	0.08557701	-0.3100192	0.7499526	0.1299141	0.04307977	0.175222450
6	0.08161734	-0.3100192	0.7499526	0.1299141	0.04108646	0.251882272
7	0.08284405	-0.3100192	0.7499526	0.1299141	0.04170399	0.287474333
8	0.07783010	-0.3100192	0.7499526	0.1299141	0.03917995	0.501026694
9	0.07882114	-0.3100192	0.7499526	0.1299141	0.03967884	0.536618754
10	0.07634669	-0.3100192	0.7499526	0.1299141	0.03843320	0.750171116
11	0.07712322	-0.3100192	0.7499526	0.1299141	0.03882410	0.785763176
12	0.07331750	-0.3100192	0.7499526	0.1299141	0.03690829	1.267624914
13	0.07003976	-0.3100192	0.7499526	0.1299141	0.03525827	1.765913758
14	0.06897968	-0.3100192	0.7499526	0.1299141	0.03472461	2.264202601

Full SVI calibration

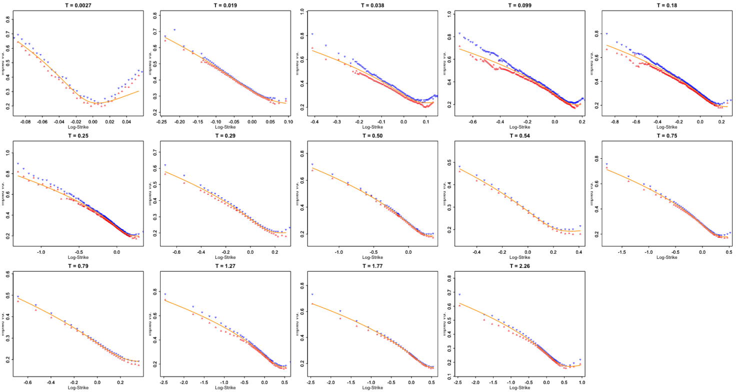


Figure 8: SPX option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the SVI fit

Full SVI calibration: March 2012 detail

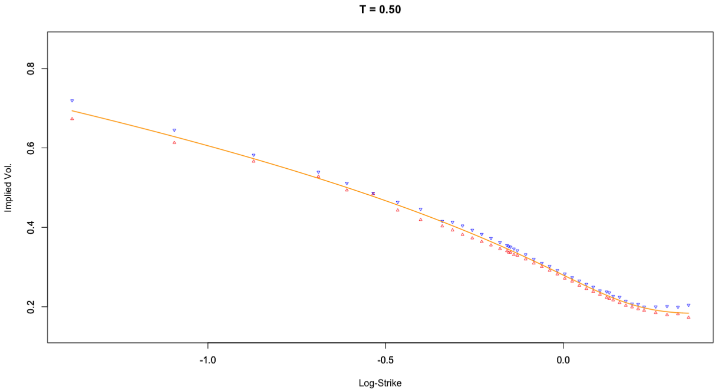


Figure 9: SPX Mar-2012 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the SVI fit

Full SVI calibration: Total variance plot

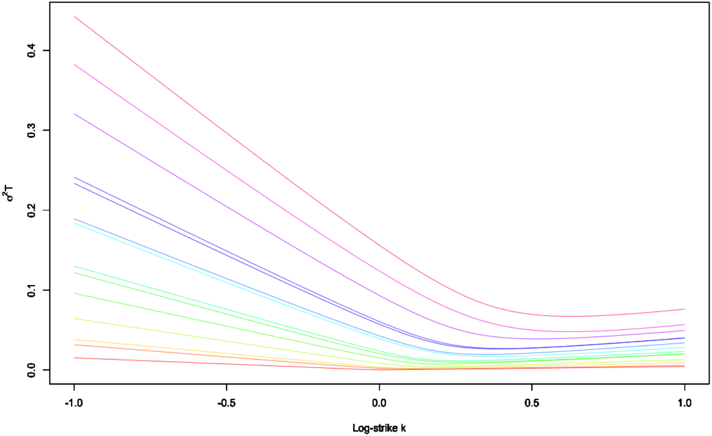


Figure 10: Total variance plot for full SVI fit: No lines cross!

Full SVI calibration: Zoomed total variance plot

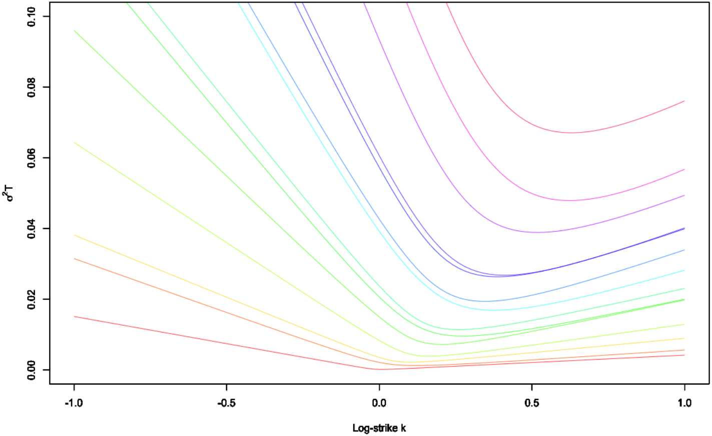


Figure 11: Total variance plot for full SVI fit: No lines cross!

Full SVI calibration: 3D plot

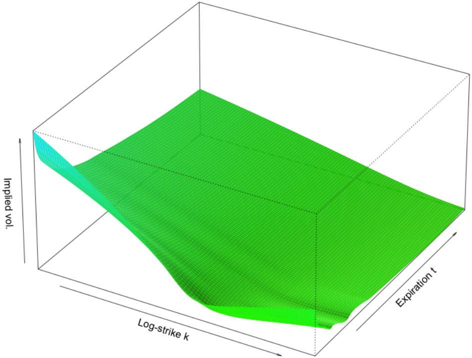


Figure 12: Fitted SPX volatility surface as of 3pm on 15-Sep-2011

Full SVI calibration: 3D plot of local variance

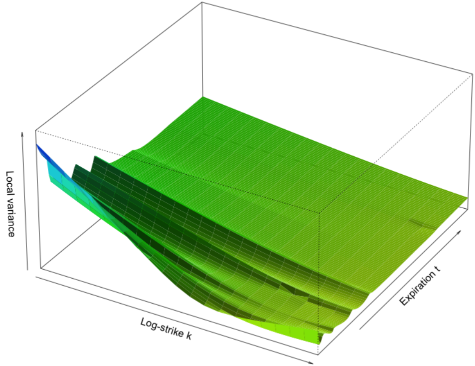


Figure 13: Fitted SPX local variance surface as of 3pm on 15-Sep-2011

JW parameters for full calibration

	vt	psit	pt	ct	varmint	texp
1	0.04964328	-0.05644814	1.3188159	0.3601864	0.04839323	0.002737851
2	0.11030822	-0.18695177	0.6669708	0.1257191	0.06419691	0.019164956
3	0.09185758	-0.22577697	0.5943167	0.1387680	0.05672638	0.038329911
4	0.08430449	-0.27032405	0.6237999	0.1327778	0.03952931	0.098562628
5	0.08538359	-0.28671259	0.6769629	0.1522872	0.04111123	0.175222450
6	0.08175423	-0.28913126	0.7311452	0.1302903	0.03800972	0.251882272
7	0.08246796	-0.29892633	0.7075543	0.1272634	0.03958896	0.287474333
8	0.07818454	-0.30641514	0.7626481	0.1349955	0.03365778	0.501026694
9	0.07939100	-0.30961650	0.7348491	0.1480980	0.03610965	0.536618754
10	0.07626063	-0.32362553	0.7630535	0.1336569	0.03510533	0.750171116
11	0.07705433	-0.32328474	0.7613835	0.1369960	0.03407266	0.785763176
12	0.07357245	-0.33341215	0.7738590	0.1244284	0.03065953	1.267624914
13	0.07010458	-0.32922668	0.7719227	0.1459205	0.02711859	1.765913758
14	0.06895374	-0.33210301	0.7537292	0.1302102	0.02960947	2.264202601

- Note that JW parameters are almost independent of texp

Full SVI calibration: ATM skew

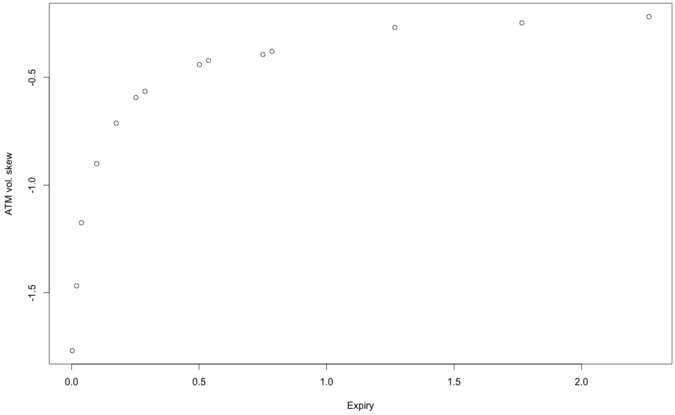


Figure 14: At-the-money volatility skew from the full calibration

Full SVI calibration: ATM skew

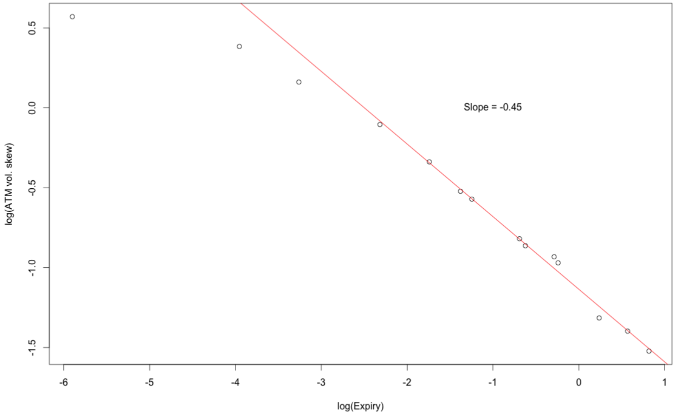


Figure 15: Log-log plot of ATM skew with regression slope

Full SVI calibration: ATM skew

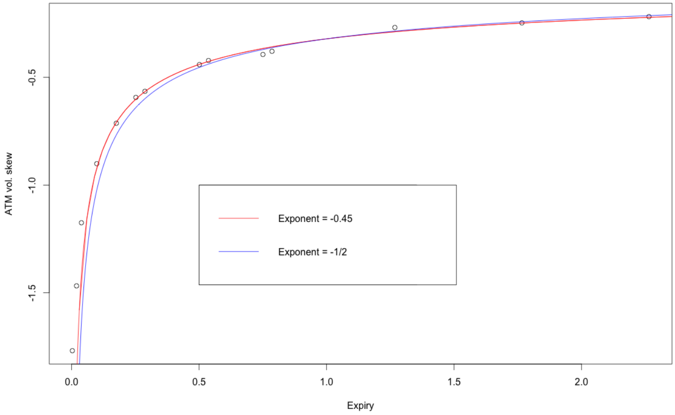


Figure 16: ATM skew again with power-law fits

Full SVI calibration: Variance swap term structure

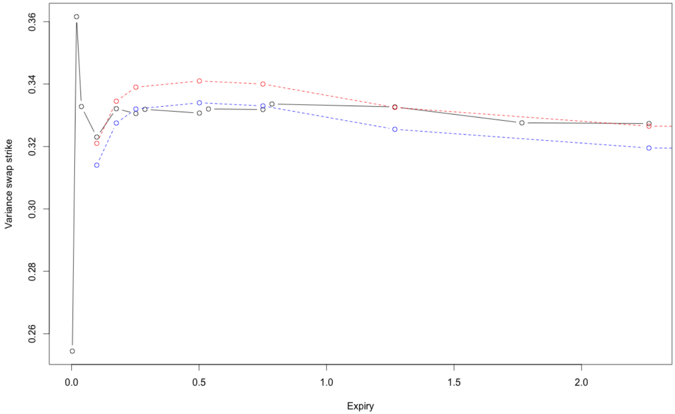


Figure 17: Market variance swaps bids and offers (in blue and red) vs the log-strip computation (in black)

SVI-SABR

- Consider the (lognormal) SABR formula with $\beta = 1$:

$$\sigma_{BS}(k) = \alpha\, f\left(\frac{k}{\alpha}\right)$$

with

$$f(y) = -\frac{\nu\, y}{\log\left(\frac{\sqrt{\nu^2\, y^2 + 2\,\rho\,\nu\, y + 1} - \nu\, y - \rho}{1 - \rho}\right)}. \tag{5}$$

- Compare this with the simpler SVI-SABR formula:

$$\sigma^2_{\mathrm{BS}}(k) = \frac{\alpha^2}{2}\left\{1 + \rho\,\frac{\nu}{\alpha}\,k + \sqrt{\left(\frac{\nu}{\alpha}\,k + \rho\right)^2 + (1 - \rho^2)}\right\} \tag{6}$$

which is guaranteed free of butterfly arbitrage if $\alpha\,\nu\,(1 + |\rho|) < 4$ and $\nu^2\,(1 + |\rho|) < 4$.

Butterfly arbitrage

- It is well known that the SABR volatility smile is susceptible to butterfly arbitrage.
    - The corresponding density is often negative for extreme strikes.
- On the other hand, the SVI-SABR density is guaranteed positive so long as $\alpha\,\nu\,t\,(1+|\rho|)<4$ and $\nu^2\,t\,(1+|\rho|)<4$.
    - Typical values of these parameters for SPX are $\nu^2\,t=0.6$, $\alpha=0.2$, $\rho=-0.7$ so for SPX there is empirically no butterfly arbitrage.
    - SABR and SVI-SABR fit parameters are not identical but they are similar.

An example: March 2012 again

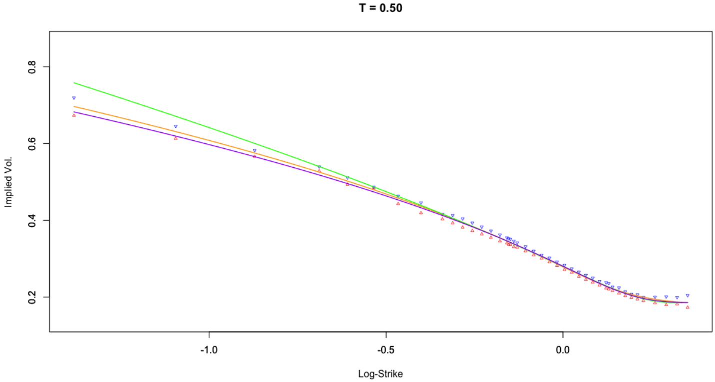


Figure 18: SPX Mar-2012 option quotes as of 3pm on 15-Sep-2011. Red and blue triangles are bid and ask implied volatilities; the orange solid line is the SVI fit, the green line the SABR fit, the purple line the SVI-SABR fit

Plots of $g(k)$

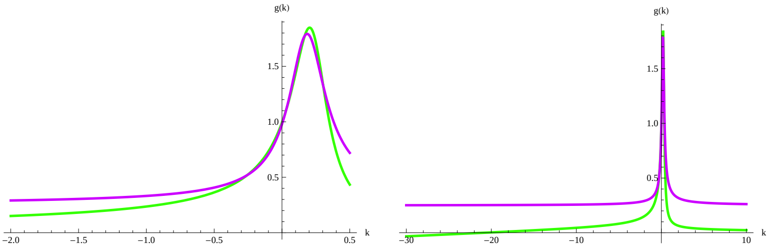


Figure 19: $g(k)$ for the SABR fit is in green, $g(k)$ for the SVI-SABR fit in purple. The negative SABR density is clearly visible in the extreme left wing.

- We note that around at-the-money, the two densities are very similar. However, as the strike moves away from ATM, the densities diverge and the SABR density goes negative.

Summary

- We have found and described a large class of arbitrage-free SVI volatility surfaces with a simple closed-form representation.
- Taking advantage of the existence of such surfaces, we showed how to eliminate both calendar spread and butterfly arbitrages when calibrating SVI to implied volatility data.
- We further demonstrated the high quality of typical SVI fits with a numerical example using recent SPX options data.
- Finally, we showed how a guaranteed arbitrage-free simple SVI smile could potentially replace SABR in applications.

References

[1] Andreasen J., Huge B. Volatility interpolation, *Risk*, 86–89, March 2011.

[2] Carr, P., Wu, L. A new simple approach for for constructing implied volatility surfaces, *Preprint* available at SSRN, 2010.

[3] Fengler, M. Arbitrage-free smoothing of the implied volatility surface, *Quantitative Finance* **9**(4): 417–428, 2009.

[4] Gatheral, J., A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives, *Presentation at Global Derivatives*, 2004.

[5] Gatheral, J., The Volatility Surface: A Practitioner's Guide, Wiley Finance, 2006.

[6] Gatheral, J., Jacquier, A., Convergence of Heston to SVI, *Quantitative Finance* **11**(8): 1129–1132, 2011.

[7] Gatheral, J., Jacquier, A., Arbitrage-free SVI volatility surfaces, *SSRN preprint*, 2012.

[8] Glaser, J., Heider, P., Arbitrage-free approximation of call price surfaces and input data risk, *Quantitative Finance* **12**(1): 61–73, 2012.

[9] Jäckel, P., Kahl, C. Hyp hyp hooray, *Wilmott Magazine* 70–81, March 2008.

[10] Kahalé, N. An arbitrage-free interpolation of volatilities, *Risk* **17**:102–106, 2004.

[11] Lee, R., The moment formula for implied volatility at extreme strikes, *Mathematical Finance* **14**(3): 469–480, 2004.

[12] Zeliade Systems, Quasi-explicit calibration of Gatheral's SVI model, *Zeliade white paper*, 2009.