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Price and Volatility Co-jumps

Pisa, 15 feb 2012

Seminar outline

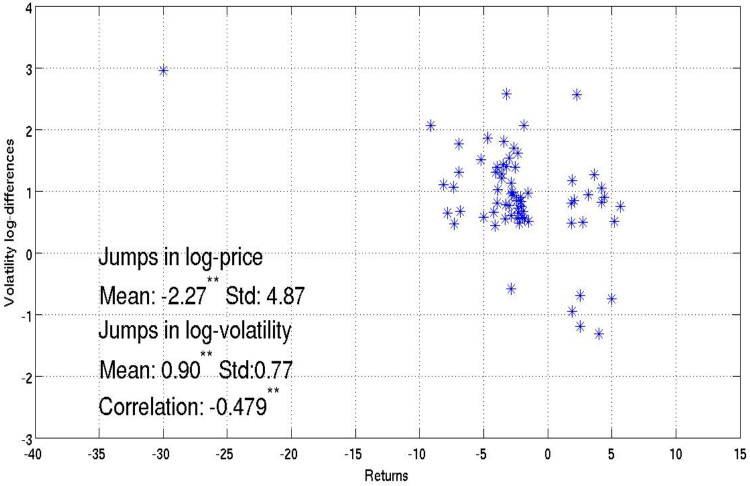
Largest price movements

Day	Return	t-stat	Volatility change	t-stat	Description
03-Aug-1984	3.63	4.28	1.26	8.8	
18-Dec-1984	3.09	3.72	0.34	2.4	
08-Jan-1986	-3.30	-4.56	0.78	5.5	
11-Sep-1986	-5.19	-5.77	1.51	10.6	
16-Oct-1987	-7.35	-6.66	1.06	7.4	
19-Oct-1987	-30.01	-24.26	2.96	20.7	Black Monday
14-Apr-1988	-4.68	-4.27	1.86	13.0	Dollar plunge
17-Mar-1989	-2.75	-4.02	0.98	6.9	
13-Oct-1989	-6.85	-10.74	0.67	4.7	Friday 13th
12-Jan-1990	-3.43	-4.37	1.81	12.6	
22-Jan-1990	-3.47	-3.95	1.42	10.0	
17-Jan-1991	4.43	4.89	0.91	6.3	
21-Aug-1991	2.74	3.99	0.03	0.2	
15-Nov-1991	-4.08	-6.57	1.30	9.1	
16-Feb-1993	-2.52	-4.78	1.38	9.7	

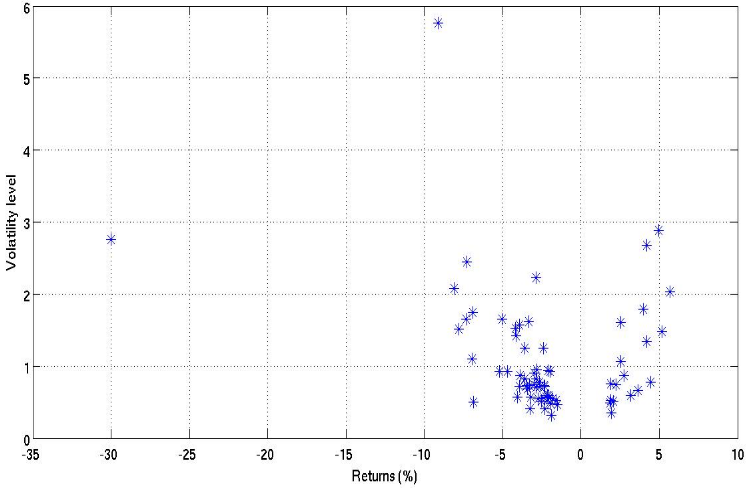
Largest price movements (continued)

Day	Return	t-stat	Volatility change	t-stat	Description
04-Feb-1994	-2.33	-5.76	1.62	11.3	
08-Mar-1996	-3.94	-4.92	1.39	9.7	
05-Jul-1996	-2.36	-3.69	0.56	3.9	
27-Oct-1997	-7.80	-7.46	0.64	4.5	Asian Crisis
28-Oct-1997	5.68	5.09	0.75	5.3	
09-Jan-1998	-3.88	-4.08	1.02	7.2	
04-Aug-1998	-3.60	-3.76	1.21	8.5	
31-Aug-1998	-7.30	-5.41	0.47	3.3	Russian crisis
04-Jan-2000	-3.52	-3.99	-0.20	-1.4	
14-Apr-2000	-8.11	-4.90	1.11	7.7	Dot.com crash
03-Jan-2001	5.18	3.90	0.50	3.5	
17-Sep-2001	-5.02	-4.22	0.58	4.0	9/11
20-Jan-2006	-1.93	-3.64	0.67	4.7	
27-Feb-2007	-3.23	-7.07	2.58	18.0	Chinese Correction
29-Sep-2008	-6.93	-4.09	1.76	12.3	Lehman-Brothers default

Scatter plot



Volatility-dependent size



Some comments

- Large price movements are typically associated with large volatility movements
- Large price movements are more often negative, while large volatility movements are almost always positive
- There appears to be a strong negative correlation between the jump sizes in price and volatility

Previous literature

- Duffie, Pan and Singleton (2000): co-jumps in a parametric affine model, strong negative correlation between jump sizes. Estimation methodology: calibration on options.
- Eraker, Johannes and Polson (2003): co-jumps in a parametric affine model with no independent jumps. Estimation methodology: MCMC on return time series.
- Eraker (2004): same as EJP. Estimation methodology: MCMC on joint return and option data
- Todorov and Tauchen (2010): nonparametric evidence on co-jumps between returns and the VIX index.

Our approach

- We use only price data (no options, no need of modelling risk premia) but we exploit the availability of intraday data to filter spot volatility estimates.
- We use a flexible nonparametric model with stochastic volatility which allows for both indipendent jumps and co-jumps
- We reconstuct the dynamics, both parametrically and non-parametrically, using a GMM approach based on infinitesimal moments, estimated with the Nadaraya-Watson approach.
- We study the asymptotic properties of the feasible estimators in which spot variances are replaced with estimated variances

The model

$$
\begin{aligned}
d(\log p_t) &= \mu(\sigma_t)dt + \sigma_t\left\{\rho(\sigma_t)dW_t^1 + \sqrt{1-\rho^2(\sigma_t)}dW_t^2\right\} \\
&\quad + c_{r,t}^J dJ_r + c_{r,t}^{JJ} dJ_{r,\sigma} \qquad (1) \\
d\xi(\sigma_t^2) &= m(\sigma_t)dt + \Lambda(\sigma_t)dW_t^1 + c_{\sigma,t}^J dJ_\sigma + c_{\sigma,t}^{JJ} dJ_{r,\sigma},
\end{aligned}
$$

where $\xi(\cdot)$ is an increasingly monotonic function, $W = \left\{W^1, W^2\right\}$ is a bivariate standard Brownian motion vector, $J = \left\{J_r, J_\sigma, J_{r,\sigma^2}\right\}$ is an independent (of W) trivariate vector of mutually independent Poisson processes with intensities $\lambda_r(\sigma_t)$, $\lambda_\sigma(\sigma_t)$, and $\lambda_{r,\sigma}(\sigma_t)$, respectively. The functions $\mu(\cdot), m(\cdot), \Lambda(\cdot), \lambda_r(\cdot), \lambda_\sigma(\cdot), \lambda_{r,\sigma}(\cdot)$ and $\rho(\cdot)$ satisfy mild smoothness conditions and are solely such that a unique, recurrent, strong solution to the system exists.

System features

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- Independent jumps and co-jumps
- State-dependent intensities
  (the state driving the dynamics is the volatility σ_t)
- State-dependent jumps
- Time-varying leverage
  (we provide evidence of more negative leverage corresponding to higher volatility levels)
- Possibly non-affine structures
  (we also provide evidence that affine models might be misspecified)

Infinitesimal cross-moments

- Assume that we work with a logarithmic variance specification ($\xi(\sigma_t^2) = \log(\sigma_t^2)$) and Gaussian jumps.
- The key element of the identification method we propose is the generic *infinitesimal cross-moment* of order p_1, p_2 with $p_1 \geq p_2 \geq 0$, namely

$$\vartheta_{p_1,p_2}(\sigma) = \lim_{\Delta \to 0} \frac{1}{\Delta} \mathbf{E} \left[(\log p_{t+\Delta} - \log p_t)^{p_1} \left(\log(\sigma_{t+\Delta}^2) - \log(\sigma_t^2) \right)^{p_2} | \sigma_t = \sigma \right]. \quad (2)$$

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- Volatility moments:

$$\vartheta_{0,1} = m + \vartheta^{Jump}_{0,1}, \quad (3)$$

$$\vartheta_{0,2} = \Lambda^2 + \vartheta^{Jump}_{0,2}, \quad (4)$$

$$\vartheta_{0,p_2} = \vartheta^{Jump}_{0,p_2} \qquad p_2 \geq 3 \quad (5)$$

with

$$\vartheta_{0,p_2}^{Jump} = \lambda_{r,\sigma} \sum_{j=0}^{p_2} \begin{pmatrix} p_2 \\ j \end{pmatrix} G_{0,j} \left(\sigma_{JJ,\sigma}\right)^j \left(\mu_{JJ,\sigma}\right)^{p_2-j} + \lambda_\sigma \sum_{j=0}^{p_2} \begin{pmatrix} p_2 \\ j \end{pmatrix} G_{0,j} \left(\sigma_{J,\sigma}\right)^j \left(\mu_{J,\sigma}\right)^{p_2-j},$$

where $G_{0,0} = 1$ and, for $g, g_1, g_2 \geq 1$,

$$\begin{aligned} G_{0,2g} &= (2g-1)!!, \\ G_{0,2g-1} &= 0, \\ G_{g_1,g_2} &= (g_1+g_2-1)\rho_J G_{g_1-1,g_2-1} \\ &\quad +(g_1-1)(g_2-1)(1-\rho_J^2)G_{g_1-2,g_2-2}. \end{aligned}$$

with

$$\vartheta_{0,p_2}^{Jump} = \lambda_{r,\sigma} \sum_{j=0}^{p_2} \left(\begin{array}{c} p_2 \\ j \end{array} \right) G_{0,j} \left(\sigma_{JJ,\sigma}\right)^j \left(\mu_{JJ,\sigma}\right)^{p_2-j} + \lambda_\sigma \sum_{j=0}^{p_2} \left(\begin{array}{c} p_2 \\ j \end{array} \right) G_{0,j} \left(\sigma_{J,\sigma}\right)^j \left(\mu_{J,\sigma}\right)^{p_2-j},$$

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For example:

$$\begin{aligned} \vartheta_{0,3} &= \lambda_{r,\sigma} \left(\left(\mu_{JJ,\sigma}\right)^3 + 2\left(\mu_{JJ,\sigma}\right)\left(\sigma_{JJ,\sigma}\right)^2 \right) \\ &\quad + \lambda_\sigma \left(\left(\mu_{J,\sigma}\right)^3 + 2\left(\mu_{J,\sigma}\right)\left(\sigma_{J,\sigma}\right)^2 \right). \end{aligned}$$

Genuine cross-moments

We have:

$$\vartheta_{1,1} = \rho\Lambda\sigma + \vartheta_{1,1}^{Jump} \qquad (6)$$

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and

$$\vartheta_{1+p_1,1+p_2} = \vartheta_{1+p_1,1+p_2}^{Jump} \qquad p_1 > 1 \text{ or } p_2 > 1 \tag{7}$$

with

$$\vartheta_{p_1,p_2}^{Jump} = \lambda_{r,\sigma} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} G_{j_1,j_2} \left(\sigma_{JJ,r}\right)^{j_1} \left(\sigma_{JJ,\sigma}\right)^{j_2} \left(\mu_{JJ,r}\right)^{p_1-j_1} \left(\mu_{JJ,\sigma}\right)^{p_2-j_2} .$$

Genuine cross-moments (continued)

The cross-moment expressions imply, for instance, that

$$\vartheta_{1,1} = \rho \Lambda \sigma + \lambda_{r,\sigma} \left(\rho_J \sigma_{JJ,r} \sigma_{JJ,\sigma} + \mu_{JJ,r} \mu_{JJ,\sigma} \right),$$

and

$$\begin{aligned}\vartheta_{2,2} &= \lambda_{r,\sigma} \{ \left(\mu_{JJ,\sigma} \right)^2 \left(\mu_{JJ,r} \right)^2 + \left(\sigma_{JJ,\sigma} \right)^2 \left(\mu_{JJ,r} \right)^2 + \left(\mu_{JJ,\sigma} \right)^2 \left(\sigma_{JJ,r} \right)^2 \\ &+ \left(1 + 2\rho_J^2 \right) (\sigma_{JJ,r})^2 (\sigma_{JJ,\sigma})^2 + 4\rho_J \mu_{JJ,r} \mu_{JJ,\sigma} \sigma_{JJ,r} \sigma_{JJ,\sigma} \}.\end{aligned}$$

Estimation

Consider a sample of T days and N intraday knots within each day. Assume availability of closing logarithmic prices ($\log p_{t,i}$) and spot volatility estimates ($\log \widehat{\sigma}^2_{t,i}$) over each day $t = 1, \ldots, T$ and each knot $i = 1, \ldots, N$.

Estimation

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The generic cross-moment estimator $\widehat{\vartheta}_{p_1,p_2}$ is defined as

$$\widehat{\vartheta}_{p_1,p_2}(\sigma) = \frac{\sum_{t=1}^{T_{days}-1} \sum_{i=1}^{N} \mathbf{K}\left(\frac{\widehat{\sigma}_{t,i}-\sigma}{h}\right) \left(\log p_{t+1,i} - \log p_{t,i}\right)^{p_1} \left(\log \widehat{\sigma}^2_{t+1,i} - \log \widehat{\sigma}^2_{t,i}\right)^{p_2}}{\Delta \sum_{t=1}^{T_{days}} \sum_{i=1}^{N_{hours}} \mathbf{K}\left(\frac{\widehat{\sigma}_{t,i}-\sigma}{h}\right)}, \tag{8}$$

that is, the frequency of price/volatility returns is daily with subsampling.

Spot variance estimates

- We use intra-daily observations for spot volatility estimation.
- We employ $N = 6$ knots in the interval $10.45am - 3.45pm$, each separated by an hour.
- Define one-minute logarithmic returns $r_{t,i,k} = \log p_{t,i,k} - \log p_{t,i,k-1}$, for $k = 1, \dots, 60$, over each hour before a knot (thus we start using observations at $9.45am$).
- The spot volatility estimates are the jump-robust TBPV:

$$\hat{\sigma}^2_{t,i} = \frac{60}{59 - n_j} \varsigma_1^{-2} \sum_{k=2}^{60} |r_{t,i,k}||r_{t,i,k-1}| I_{\{|r_{t,i,k}| \le \theta_{t,i,k}\}} I_{\{|r_{t,i,k-1}| \le \theta_{t,i,k-1}\}}, \tag{9}$$

where $\varsigma_1 \simeq 0.7979$.

Theory

Theorem 1. (Consistency.) *If* $n, T \to \infty$ *and* $\Delta_{n,T} = T/n \to 0$ *so that* $h_{n,T}\widehat{L}_{n,T}(y) \overset{a.s.}{\to} \infty$ *and* $\frac{\Delta_{n,T}}{h^2_{n,T}} \to 0$, *then*

$$\widehat{\vartheta}_{p_1,0}(y) \overset{p}{\to} \begin{cases} \mu_X(y) + \lambda_X(y)\mathrm{E}[c_X] + \lambda_{XY}(y)\mathrm{E}[d_X] & p_1 = 1 \\ \sigma^2_X(y) + \lambda_X(y)\mathrm{E}[c^2_X] + \lambda_{XY}(y)\mathrm{E}[d^2_X] & p_1 = 2 \\ \lambda_X(y)\mathrm{E}[c^{p_1}_X] + \lambda_{XY}(y)\mathrm{E}[d^{p_1}_X] & p_1 \geq 3 \end{cases},$$

$$\widehat{\vartheta}_{1,1}(y) \overset{p}{\to} \rho(y)\sigma_X(y)\sigma_Y(y) + \lambda_{XY}(y)\mathrm{E}[d_X d_Y],$$

and, without loss of generality, for $p_1 \geq p_2 \geq 1$ *(with* $p_1 > p_2$ *if* $p_2 = 1$*),*

$$\widehat{\vartheta}_{p_1,p_2}(y) \overset{p}{\to} \lambda_{XY}(y)\mathrm{E}[d^{p_1}_X d^{p_2}_Y].$$

Theorem 2. (Weak convergence.) *Let* $n, T \to \infty$ *and* $\Delta_{n,T} = T/n \to 0$ *so that* $h_{n,T}\widehat{L}_{n,T}(y) \stackrel{a.s.}{\to} \infty$ *and* $\frac{\Delta_{n,T}\sqrt{\widehat{L}_{n,T}(y)}}{h_{n,T}^{3/2}} \stackrel{a.s.}{\to} 0$. *If*

$$h_{n,T}^5 \widehat{L}_{n,T}(y) = O_{a.s.}(1),$$

then

$$\sqrt{h_{n,T}\widehat{L}_{n,T}(y)} \left\{ \widehat{\vartheta}_{p_1,p_2}(y) - \vartheta_{p_1,p_2}(y) - \Gamma_{\vartheta_{p_1,p_2}}(y) \right\} \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \vartheta_{2p_1,2p_2}(y)),$$

with

$$\Gamma_{\vartheta_{p_1,p_2}} = h_{n,T}^2 \mathbf{K}_1 \left(\frac{\partial \vartheta_{p_1,p_2}(y)}{\partial y} \frac{\frac{\partial s(y)}{\partial y}}{s(y)} + \frac{1}{2} \frac{\partial^2 \vartheta_{p_1,p_2}(y)}{\partial^2 y} \right),$$

where $s(dx)$ *is the invariant measure of the* Y *process.*

Theorem 3. *Write* $\Psi_{n,k,\phi} = \sqrt{\frac{\log(n)}{k}} + \sqrt{\phi}$. Consider $\widehat{\vartheta}_{p_1,p_2}(.)$ with $(p_1, p_2) = (1, 0)$ or $(0, 1)$. *If*

$$\frac{\Psi_{n,k,\phi}}{\Delta_{n,T}} \to 0,$$

the consistency result in Theorem 1 holds when replacing $\sigma^2_{iT/n}$ *with* $\widehat{\sigma}^2_{iT/n}$. *For any other combination of* (p_1, p_2)*, if*

$$\frac{\Psi_{n,k,\phi}}{\Delta^{1/2}_{n,T} h_{n,T}} \to 0,$$

the consistency result in Theorem 1 holds when replacing $\sigma^2_{iT/n}$ *with* $\widehat{\sigma}^2_{iT/n}$. *Assume* $(p_1, p_2) = (1, 0)$ or $(0, 1)$, *if*

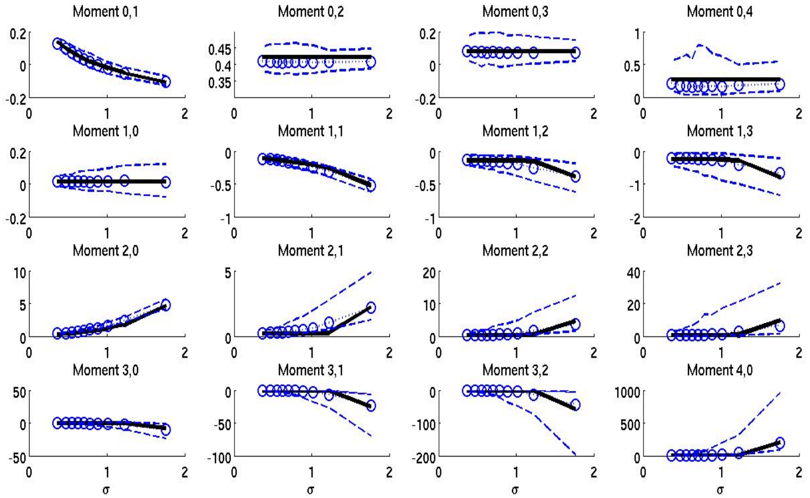
$$\sqrt{h_{n,T}\widehat{\overline{L}}_{\sigma^2}(T, \sigma^2)}\frac{\Psi_{n,k,\phi}}{\Delta_{n,T}} \to 0,$$

where $\widehat{\overline{L}}_{\sigma^2}(T, \sigma^2)$ *is the estimated occupation density of spot variance process, the weak convergence results in Theorem 2 holds when replacing* $\sigma^2_{iT/n}$ *with* $\widehat{\sigma}^2_{iT/n}$. *For any other combination of* (p_1, p_2)*, if*

$$\sqrt{h_{n,T}\widehat{\overline{L}}_{\sigma^2}(T, \sigma^2)}\frac{\Psi_{n,k,\phi}}{\Delta^{1/2}_{n,T} h_{n,T}} \to 0,$$

the weak convergence results in Theorem 2 holds when replacing $\sigma^2_{iT/n}$ *with* $\widehat{\sigma}^2_{iT/n}$.

Simulations



A GMM approach

- The infinitesimal cross-moments introduced here lend themselves to an estimation method akin to pointwise GMM (Hansen, 1982).
- Denote by $g_1(\sigma), \ldots, g_K(\sigma)$ the K functions driving the dynamics of the system.
- Consider a set of N cross-moments $\widehat{\vartheta}_{p_1,p_2}(\sigma)$ with $N \geq K$ for identification.
- The theoretical cross-moments $\vartheta_{p_1,p_2}(\sigma) = f_{p_1,p_2}(g_1(\sigma), \ldots, g_K(\sigma))$ are a mapping f_{p_1,p_2} from the functions $g_1(\sigma), \ldots, g_K(\sigma)$.
- For every value σ in the spot volatility range, the K vector of estimates $(\widehat{g}_1(\sigma), \ldots, \widehat{g}_K(\sigma))$ is defined as:

$$(\widehat{g}_1(\sigma), \ldots, \widehat{g}_K(\sigma)) = \underset{(g_1(\sigma),\ldots,g_K(\sigma))}{\arg\min} (\widehat{\vartheta}_{p_1,p_2}(\sigma) - \vartheta_{p_1,p_2}(\sigma))^\top W(\sigma)(\widehat{\vartheta}_{p_1,p_2}(\sigma) - \vartheta_{p_1,p_2}(\sigma))$$

A GMM approach: the parametric case

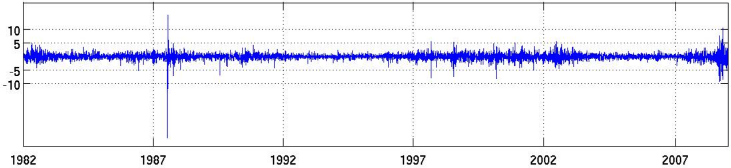
- Assume now to have a parametric specification of the main model.
- Denote by η a vector of *M* parameters.
- Select a number *G* of knots $\sigma_1, \ldots, \sigma_G$, so that $N \times G \geq M$ for identification.
- Denote by $\widehat{\vartheta}_{p_1,p_2}$ the $N \times G$-vector of available estimated moments computed at the knots σ_i with $i = 1, ..., G$ and by $\vartheta_{p_1,p_2}(\eta)$ the corresponding $N \times G$-vector of theoretical moments.
- The parametric estimates are now given by:

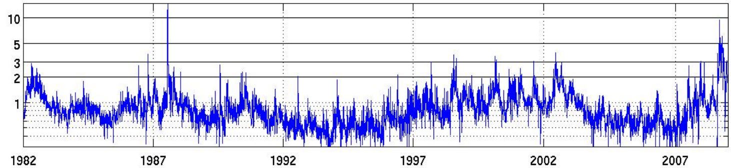
$$\widehat{\eta} \;=\; \arg\min_{\eta}(\widehat{\vartheta}_{p_1,p_2} - \vartheta_{p_1,p_2}(\eta))^\top W(\sigma)(\widehat{\vartheta}_{p_1,p_2} - \vartheta_{p_1,p_2}(\eta))$$

where $W(\sigma)$ is an $(N \times G) \times (N \times G)$ symmetrical and positive definite weighting matrix.

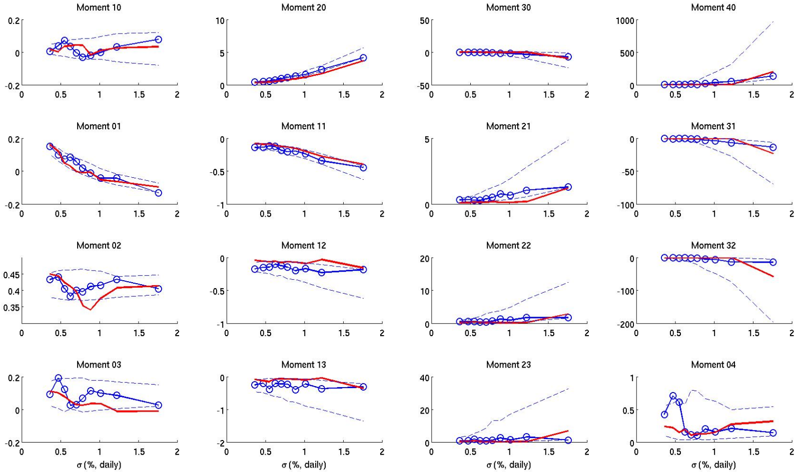
Data

Dataset: all transactions on S&P 500 futures from from April 21, 1982, to February 5, 2009, for a total of $6,748$ trading days.

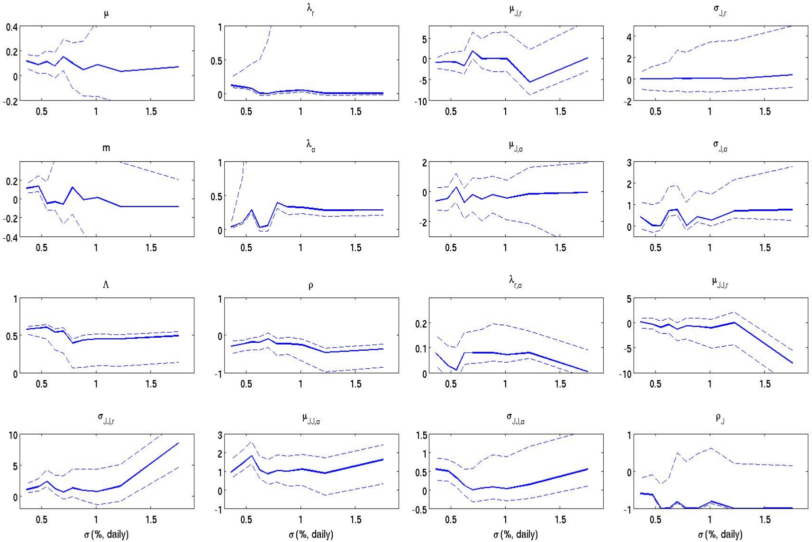




Estimated infinitesimal moments



Estimated functions



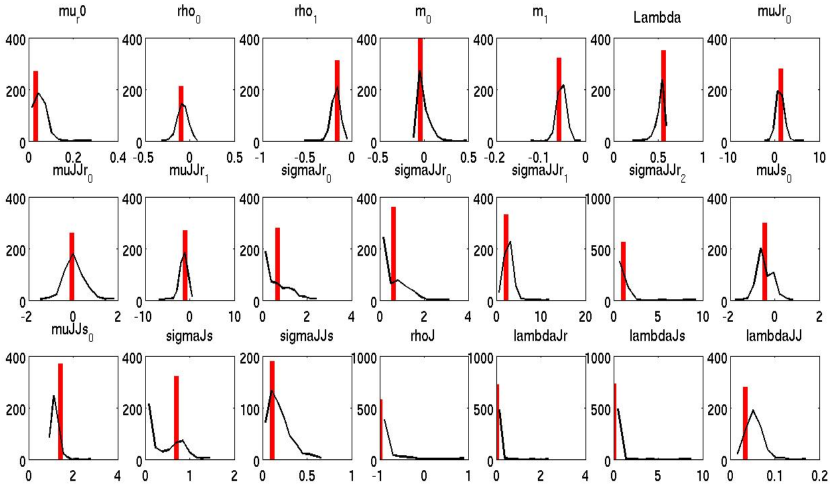
A parametric model

$$
\begin{aligned}
d\log p_t &= \mu_r dt + \sigma_t \left\{ \rho_t dW_t^1 + \sqrt{1-\rho_t^2} dW_t^2 \right\} + c_{r,t}^J dJ_r + c_{r,t}^{JJ} dJ_{r,\sigma} \\
d\log(\sigma_t^2) &= \left(m_0 + m_1 \log(\sigma_t^2)\right) dt + \Lambda dW_t^1 + c_{\sigma,t}^J dJ_\sigma + c_{\sigma,t}^{JJ} dJ_{r,\sigma}, \\
\rho_t &= \max(\min(\rho_0 + \rho_1 \sigma_t, 1), -1), \\
\{J_r, J_\sigma, J_{r,\sigma}\} &\sim Poisson(\lambda_r, \lambda_\sigma, \lambda_{r,\sigma}) \\
c_{r,t}^J &\sim \mathcal{N}\left(\mu_{J,r}, \sigma_{J,r}^2\right) \\
c_{\sigma,t}^J &\sim \mathcal{N}\left(\mu_{J,\sigma}, \sigma_{J,\sigma}^2\right) \\
\begin{pmatrix} c_{r,t}^{JJ} \\ c_{\sigma,t}^{JJ} \end{pmatrix} &\sim \mathcal{N}\left(\begin{pmatrix} \mu_{JJ,r,0} + \mu_{JJ,r,0}\sigma_t \\ \mu_{JJ,\sigma} \end{pmatrix}, \right. \\
& \left. \begin{pmatrix} \left(\sigma_{JJ,r,0} + \sigma_{JJ,r,1}\sigma_t^{\sigma_{JJ,r,2}}\right)^2 & \rho_J \left(\sigma_{JJ,r,0} + \sigma_{JJ,r,1}\sigma_t^{\sigma_{JJ,r,2}}\right)\sigma_{JJ,\sigma} \\ \blacklozenge & \sigma_{JJ,\sigma}^2 \end{pmatrix} \right).
\end{aligned}
$$

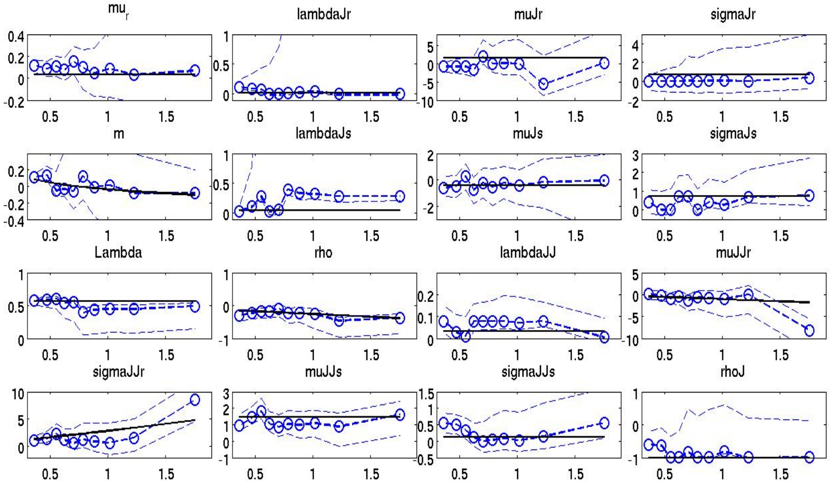
Estimates

parameter	no cojumps	no independent jumps	with cojumps	
μ_r	0.0423	0.0631	0.0306	(0.0000, 0.1110)
ρ_0	−0.2280	−0.0977	−0.0988	(−0.2073, 0.0323)
ρ_1	−0.0874	−0.1225	−0.1617	(−0.2901, −0.0863)
m_0	−0.0232	−0.0397	−0.0380	(−0.0853, 0.1581)
m_1	−0.0704	−0.0576	−0.0597	(−0.0710, −0.0347)
Λ	0.6048	0.5950	0.5583	(0.3933, 0.5830)
$\mu_{J,r}$	−0.1137	–	1.3948	(−0.4916, 3.0597)
$\mu_{JJ,r,0}$	–	0.5210	−0.0544	(−0.9454, 1.1129)
$\mu_{JJ,r,1}$	–	−1.8976	−1.0072	(−4.3072, 0.0713)
$\sigma_{J,r}$	1.2715	–	0.6818	(0.0000, 1.9688)
$\sigma_{JJ,r,0}$	–	1.7428	0.6246	(0.0000, 1.7976)
$\sigma_{JJ,r,1}$	–	0.1718	2.2469	(0.8738, 4.8970)
$\sigma_{JJ,r,2}$	–	1.8828	1.0863	(0.5747, 2.0345)
$\mu_{J,\sigma}$	0.3498	–	−0.4497	(−1.0585, 0.2008)
$\mu_{JJ,\sigma}$	–	0.7816	1.4428	(0.9511, 1.5641)
$\sigma_{J,\sigma}$	1.2575	–	0.7002	(0.0002, 1.0957)
$\sigma_{JJ,\sigma}$	–	0.4901	0.1084	(0.0105, 0.5329)
ρ_J	–	−0.6416	−1.0000	(−1.0000, −0.1785)
λ_r	0.1033	–	0.0252	(0.0045, 0.3052)
λ_σ	0.0279	–	0.0528	(0.0127, 0.8920)
$\lambda_{r,\sigma}$	–	0.0489	0.0339	(0.0203, 0.0978)

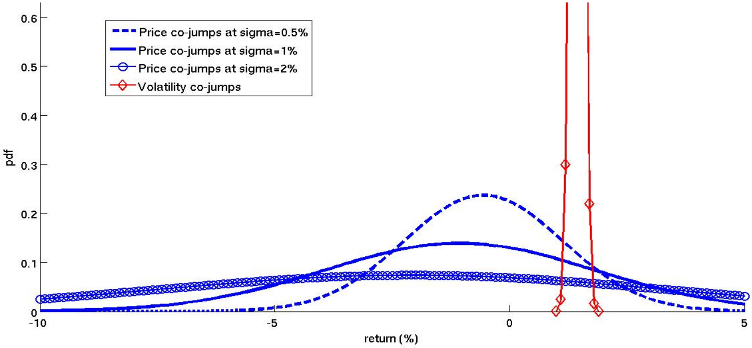
Simulations



Parametric fitting



Distribution of co-jumps



The leverage effect

$$\rho_{total} = \frac{\vartheta_{1,1}}{\sigma\Lambda} = \rho + \frac{\lambda_{r,\sigma}\left(\rho_J \sigma_{JJ,r}\sigma_{JJ,\sigma} + \mu_{JJ,r}\mu_{JJ,\sigma}\right)}{\sigma\Lambda} = \rho + \rho_{co-jumps}.$$

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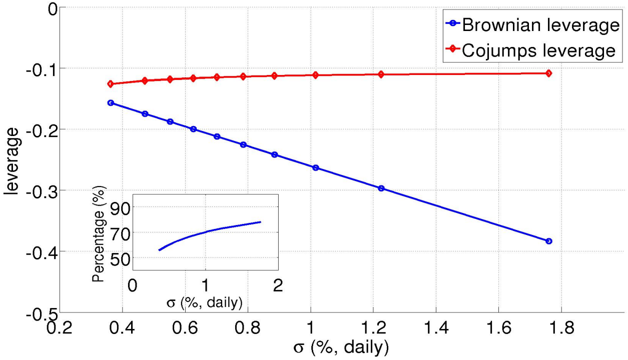
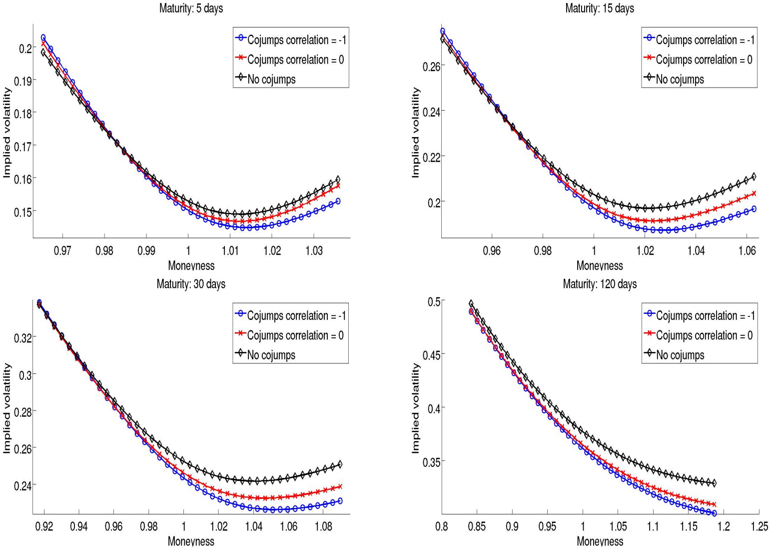


Implications for option pricing

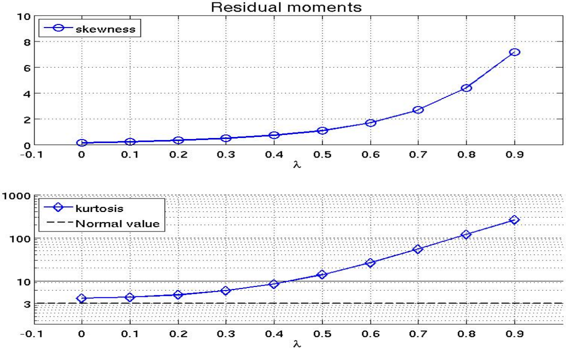


The volatility transformation

Define the system residuals as:

$$\varepsilon_{t,t+\Delta} = \frac{f_\lambda(\sigma^2_{t+\Delta}) - f_\lambda(\sigma^2_t) - m_\lambda(\sigma_t)\Delta}{\Lambda_\lambda(\sigma_t)\sqrt{\Delta}},$$

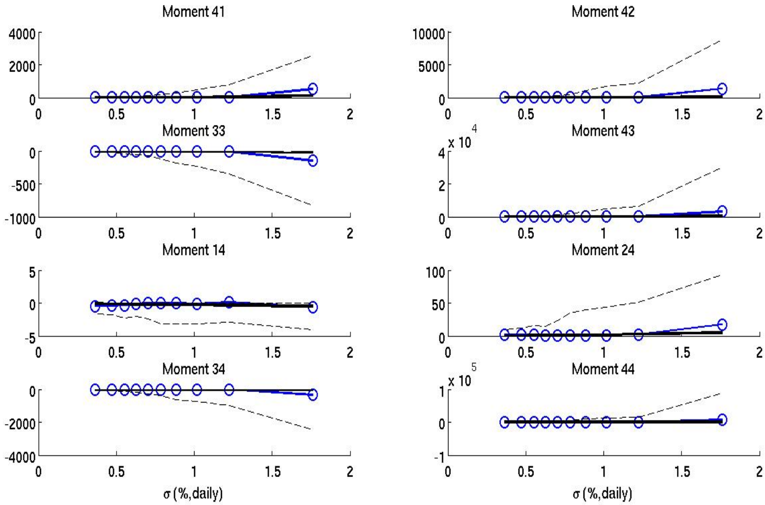
where $f_\lambda(\cdot)$ is a Box-Cox transformation.



Testing for co-jumps

test	value	p-value
J-test	239.4	0.20%
$< \vartheta_{2,2} >$	0.2862	0.10%
$< \vartheta_{1,2} >$	-0.0681	0.00%
$< \vartheta_{2,1} >$	0.2669	0.00%
$< \vartheta_{1,3} >$	-0.1091	0.00%
$< \vartheta_{3,1} >$	-0.9861	0.20%
$< \vartheta_{2,3} >$	0.4846	0.10%
$< \vartheta_{3,2} >$	-2.3185	0.10%
$< \vartheta_{3,3} >$	-6.2580	0.10%

Overidentifying restrictions



Conclusions

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- We also uncover important dynamical features of the data, such as time-varying leverage and nonlinear jump sizes
- We propose a novel approach for estimation of a dynamical (parametric) system, which is based on spot variance estimation and infinitesimal GMM.