Square-root price impact and critical liquidity

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Paper available:

Phys Rev X 1, 021006 (2011);

Outline

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Price impact

What is price impact?

- Price impact: Correlation between the sign of an arriving order and the subsequent price change.
- In general, buyer (seller) initiated trades push the price up (down).

Several types of price impact

- Single trade impact vs. impact of several connected trades.
- Immediate impact vs. longer time impact
- Impact of the average order flow.

Metaorders

A metaorder is a **series** of connected trades in the same direction from the same initiator.

Reason: when trading a large position, typically we have to **split** it up and trade incrementally.

- Not enough liquidity available at any given moment: for a liquid stock the instantaneous volume in the order book is $\approx 10^{-5}$ of market capitalisation, while the total volume traded in a day is $\approx 10^{-3}$
- Not to disseminate too much information (in case of LOs)

Knowing trading IDs, we can measure it. Evidence for order splitting can be found from trade sign correlations.

Impact of metaorders

Widely observed result

A metaorder of size Q has a price impact:

$$I(Q) = Y\sigma_T \left[\frac{Q}{V_T}\right]^{\delta}, \quad (1)$$

Q is the volume of the metaorder

σ_T is the volatility of the market

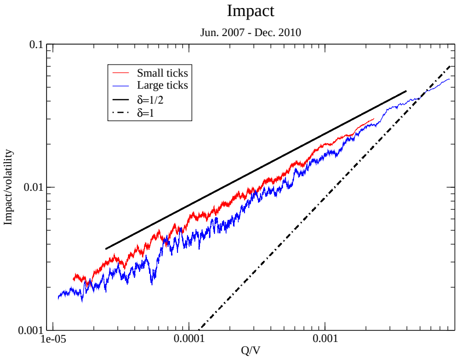
V_T is the total volume traded in the market

$\delta \in [0.4, 0.7]$; Y is a constant of order unity

Remarkable stability of results (style of trading, strategies, markets, periods, tick sizes, treatment of data..)

$$I(Q) = Y\sigma_T \left[\frac{Q}{V_T}\right]^{\delta}$$

Results from CFM trades:



Note: singular for $Q \to 0$! Trading 1% of daily volume moves the price by 10% of daily volatility!

$$I(Q) = Y\sigma_T\left[\frac{Q}{V_T}\right]^{\delta}$$

Importance of the concavity

- crucial in controling cost of trading
- concave is not additive ($I(Q_1+Q_2) \neq I(Q_1)+I(Q_2)$): but we are small compared to the market ($Q \ll V_T$), how can the response be non-linear?
- diverging susceptibility for small volumes: small orders can generate large responses
- the relation is invariant to execution time: for diffusive prices $\sigma_t \propto \sqrt{N_t}$ and $V_t \propto N_t$ (N_t: number of trades)
- also the increase of impact in volume time is square-root: $I(q) \propto I(Q)(q/Q)^{\delta}$

Theories for a concave impact

Many different ideas

1. Risk-reward of market makers (Barra 1997, Gabaix et al. 2003, etc.)
2. Detection of large metaorders, efficiency and fair-pricing (Farmer, Lillo et al. 2010)
3. Linear supply/demand profiles (old idea)

1. Risk-reward of market makers

- A market maker with inventory Q needs a time $t = TQ/V_T$ to get rid of it
- His risk is therefore $\sigma_T \sqrt{t/T}$, which should also be his reward
- Therefore he raises the price by $Y\sigma_T \sqrt{Q/V_T}$!!

Problems: i) Diversification should lower the cost; ii) Market maker can match trades internally

Concavity and linear profiles: intuition

After having traded $Q/2$, the next $Q/2$ will have less impact
$\Rightarrow$ this means that there is increasing volume available deeper in the book
However the typical order book that one **observes** is not like that.

There has to be a **latent** volume that only appears when we push the price.

Suppose the latent volume grows linearly with depth:
$V_{\pm}(p) \propto |p - p_0|$

$$Q = \int_{p_0}^{p_0+I} V_{+}(p)dp \rightarrow p = p_0 + a\sqrt{Q} \quad (2)$$

and note that $V_{\pm}(p)(p \rightarrow p_0) \rightarrow 0$

But why should the profile be linear? And linear around what price?

Outline

The idea of a latent order book

Assumptions

- between time t and $t+dt$, when price is $p(t)$, some investors decide to put a **latent** limit order at a price $p(t) \pm u$ with probability $\lambda_{\mp}(u)dt$ and with unit volume

  $\Rightarrow$ these orders are latent, since they may only actually appear later when the price approaches $p(t) \pm u$
- investors can also cancel their latent orders: $\nu_{\pm}(u)$
- the price, $p(t)$, fluctuates due to the effect of market orders: σ

$\Rightarrow$ it is this latent order book that we want to study

Basics of the model

To illustrate the calculations I simplify

Let's assume that

- limit orders arrive with a **constant, flat** rate per unit time per unit price over an infinite support ($\lambda_{\pm}(u) \equiv \lambda$)
  $\Rightarrow$ no u or volume dependence
- cancellation: **constant, flat** rate of limit orders being removed per unit time ($\nu(u) \equiv \nu$) $\Rightarrow$ defines a lifetime
- market orders: rate μ per unit time
- discretization: tick size and unit volumes

However all this is not crucial at all!

Analytical treatment

Let $\rho(u)$ be the density of the book at a distance u away from the best price. Let's assume that the price process is **diffusive**.
In the (moving) reference frame of the mid price:

$$\frac{\partial \rho}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \rho}{\partial u^2} + \lambda - \nu\rho \tag{3}$$

$$\rho = \frac{\lambda}{\nu}\left(1 - e^{-u/u^*}\right) \tag{4}$$

For small distances $(u \longrightarrow 0)$, we get

$$\rho(u) \approx \frac{\lambda}{\nu u^*} u. \tag{5}$$

The book is expected to be **linear around the current price** and **vanishing liquidity around the best**.

Analytical treatment

Linearity of the book

Again: the linearity of the book is an analytical result, **not an assumption!**
Generic behaviour, valid whenever rates are regular around $u = 0$, provided prices are diffusive!

The linear regime extends roughly up to a distance u^*, which is of the order of the price variation on the scale of the lifetime of an order. [$\approx$ daily volatility]

This is where **latent** volume becomes important: the lifetime of true liquidity (intentions) is long. These players are not sensitive to price changes much smaller than the daily volatility

But in the these analytic calculations the diffusivity was **assumed**!

The idea again

Latent volume only appears when the price gets close.
The visible book has an other shape. But the impact is determined by the total liquidity (latent + visible).

What is “latent”, what is “visible”?

Clearly trades have to be visible: they appear in the **real** order book.

- But in the model trade signs at this point are not correlated!
  $\Rightarrow$ we need to add autocorrelation in the direction of trades
- Increasing volume away from the best prices: the basic model is mean reverting, prices are subdiffusive.
  $\Rightarrow$ we need to obtain a diffusive price process

$\Rightarrow$ Some further ingredients are needed in a numerical model.

Further ingredients

Autocorrelation

LMF model[a]:

Generate runs of $+/-$ signs, with length distribution $P(L) = \alpha L^{-(\alpha+1)}$.

Can be shown that this leads to an autocorrelation *as observed empirically*:

$$C(\ell) \propto \ell^{-\gamma}, \text{ where } \gamma = \alpha - 1 \quad (6)$$

We have **long-memory** of the signs for $1 < \alpha < 2$, and provides a realistic background.

[a] F. Lillo, Sz. Mike, J.D. Farmer, Phys. Rev. E 71, 066122 (2005)

But with unit volume execution, deep and slow markets are always **subdiffusive**!

Further ingredients

Conditioning on volume

For real markets: it is the **fraction** of volume taken from the opposite best, that is (roughly) constant.
To mimic this behaviour we define the following probability of the fraction (f) being taken:

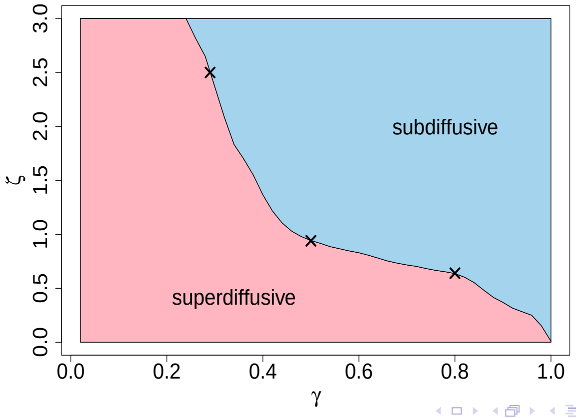
$$P(f) = \zeta(1-f)^{\zeta-1} \tag{7}$$

- $\zeta \to 0$: all volume is taken
- $\zeta = 1$: flat distribution of volume taken
- $\zeta \to \infty$: minimum (unit) volume is taken

Note: this way the best bid and ask levels are visible!

Diffusion "map"

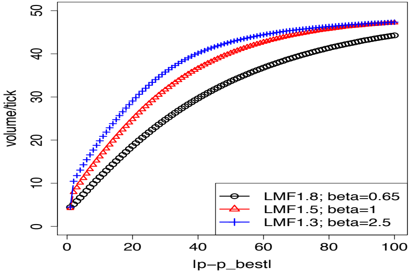
Subdiffusive vs. superdiffusive regions
"Efficient" (diffusive) boundary (what is the exact shape?)
Small ζ, small γ favour superdiffusion.



Summary of the model

A model, that aims at describing the latent liquidity. We have:

- long-memory of trade signs
- opposing flow of limit orders
- diffusive prices
- as a **result**: on average linear book



Outline

Introducing metaorders

Now we have the background model, we can introduce metaorders to study their impact.
We add an extra agent to the market: he becomes active at an arbitrary time and executes a metaorder of random sign (ε) and random size (Q).

The agent

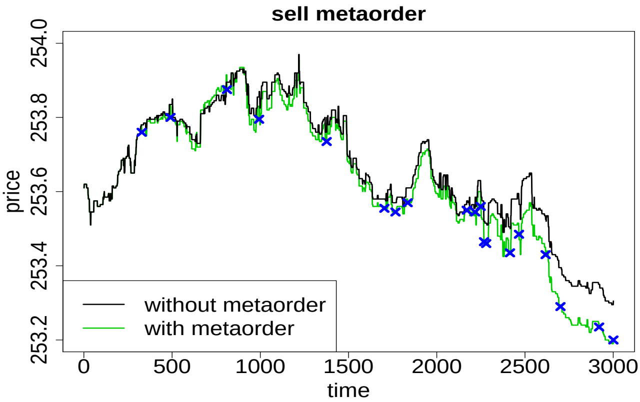
- executes incrementally
- for now, only uses MOs.

We test two execution styles

- ζ-exec: like the rest of the market
- unit-exec: executing unit volume in each trade

Introducing metaorders

Illustration: two simulations with the same random seeds, without and with metaorders



Impact of metaorders

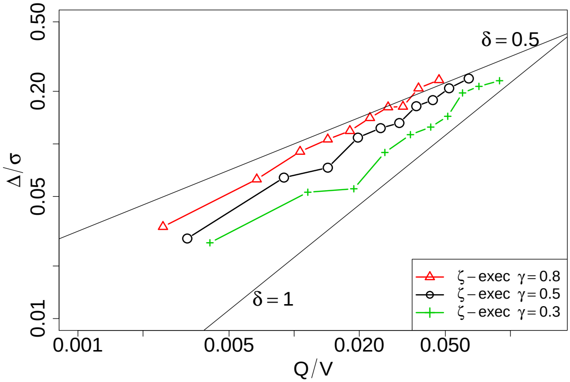


Figure: The impact for different values of γ exponent — ζ-exec.

Impact of metaorders

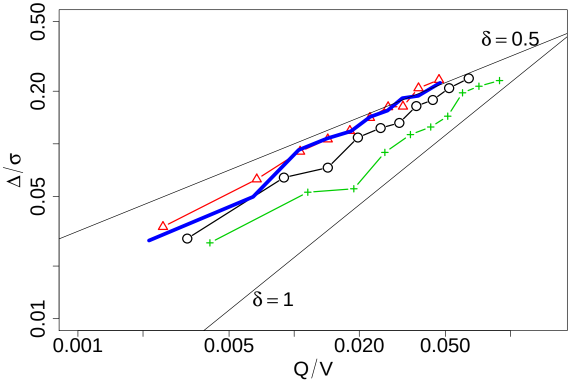


Figure: Decreasing the participation rate (from 30% to 2%).

Impact of metaorders

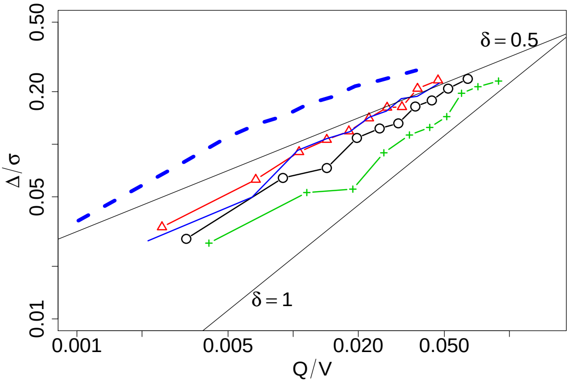


Figure: Looking at other type of execution style — unit-exec.

Impact of metaorders

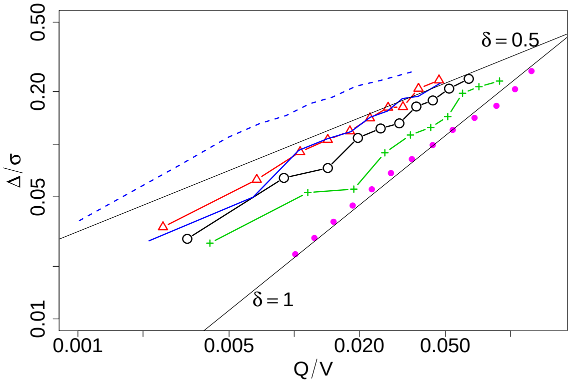


Figure: $T > \tau_{\mathrm{life}}$: If the lifetime of orders is too short: back to linear.

Impact of metaorders

For realistic parameters

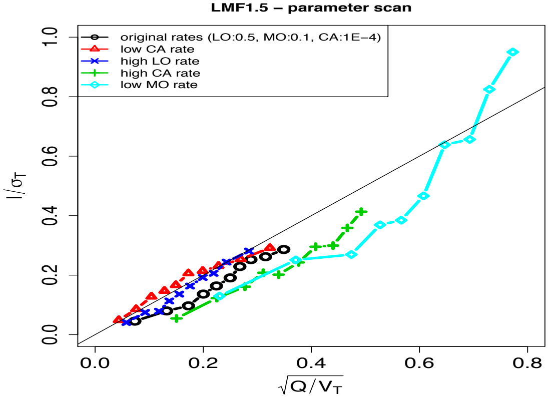
- the model reproduces the concave impact

$$I(Q) = Y\sigma_T \left[\frac{Q}{V_T}\right]^{\delta}, \qquad \delta \in [0.5, 0.7]$$

- weak dependence on the autocorrelation exponent
- no dependence on participation rate (went down to 2% participation rate, sqrt is measurable)
- Y-ratio is of order unity (between $0.8 - 1.1$)
- if lifetime of orders is too short: not enough memory $\Rightarrow$ we get back the linear impact

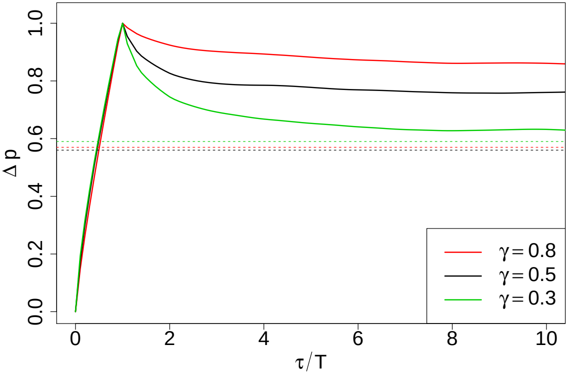
Stability of the results

Varying the parameters λ, μ, and ν.
When diffusive prices can be achieved and $\tau_{execute} << \tau_{life}$, we always find the same form.



Decay of impact

After finishing a metaorder, our impact decays. The **permanent level** depends on the γ parameter: **no "fair pricing"** (but as a coincidence...). Similar result seen for data by Moro et al. (2010)?



Outline

Summary

- Empirical fact: impact of metaorders is a concave function of volume
- Evidence for latent order book

The model

- Latent volume taken into account
- Very few behavioural assumptions (no fundamental price, no market makers, no adverse selection)
- Diffusive prices
- Analytic calculations tell that on average the book should be linear around the current price

Summary

Results

- Concave impact (when diffusive prices and $\tau_{exec} << \tau_{life}$)
- Y-ratio of order 1
- No participation rate dependence
- Stable results against changing parameters
- More realistic execution: limit orders also used $\rightarrow$ also concave, lower Y

Conclusions

The critical nature of liquidity

- Local liquidity is vanishingly small by necessity! (eaten by the diffusive price motion)
- This imposes a splitting up of metaorders and long-range memory in the sign of trades...
- ...and leads to a breakdown of linear response and an anomalously large impact for small trades ($\Rightarrow$ concave impact)
- Liquidity fluctuations are bound to play a crucial role! $\Rightarrow$ microcrises and jumps in prices