Fourier Volatility Estimation Method: Theory and Applications with High Frequency

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Motivation

- Computation of **volatility/covariance** of financial asset returns plays a central role for many issues in finance: **risk management**, hedging strategies, forecasting...
- Black&Scholes model - **constant volatility** - does not account for: heteroschedasticity, predictability, volatility smile, covariance between asset returns and volatility (leverage effect) $\Rrightarrow$
stochastic volatility models proposed to model asset price evolution and to price options (adding **risk factors** represented by Brownian motions [Heston, 1993, Hull and White, 1987, Stein and Stein, 1991], **jumps** [Bates, 1996], or introducing **memory** [Hobson and Rogers, 1998])
- Availability of **high frequency data** have the potential to improve the capability of computing volatility/covariances in an efficient way to many extend
[Andersen, Bollerslev and Meddahi, 2006] (forecasting),
[Bollerslev and Zhang, 2003] (risk factor models),
[Fleming, Kirby and Ostdiek, 2003] (asset allocation)....

Volatility: The problem

Volatility is **not observable**

Estimation

- **parametric**: the expected volatility is modelled through a functional form of variables observed in the market
- **non-parametric**: the computation of the historical volatility without assuming a functional form of the volatility

Outline

- Definition of Fourier estimator of spot and integrated volatility/covariance
- Properties of Fourier estimator with high frequency data
- Potentiality of Fourier estimator for some applications:
    - Quarticity estimation *forthcoming Quantitative Finance*
    - Volatility of Volatility and Leverage estimation *IJTAF 2010*

Forecasting Volatility *Quantitative Finance, 2011*

Contingent claim pricing-hedging (i.e. stochastic derivation of volatility along the time evolution) *Mathematical Finance, 2003*, *Malliavin-Thalmaier book, 2005*

Non-parametric calibration of the geometry of the Heath-Jarrow-Morton interest rates dynamics ($\Rightarrow$ measure of hypoellipticity of the infinitesimal generator) *Japanese Journal of Math., 2007*

Recent studies

Comparing correlation matrix estimators via Kullback-Leibler divergence, by Mattiussi, Tumminello, Iori, Mantegna

VaR/CVaR Estimation under Stochastic Volatility Models, by Liu, Han, Chen

Non-parametric and model free context

Model: continuous Brownian semimartingale

$$(\mathbf{B}) \qquad dp^j(t) = \sum_{i=1}^{d} \sigma_i^j(t)\ dW^i + b^j(t)\ dt, \qquad j = 1, \ldots, n,$$

$W = (W^1, \ldots, W^d)$ are independent Brownian motions and σ_*^* and b^* are adapted random processes satisfying

$$E[\int_0^{2\pi} (b^j(t))^2 dt] < \infty, \quad E[\int_0^{2\pi} (\sigma_i^j(t))^4 dt] < \infty \qquad i = 1, \ldots, d,\ j = 1, \ldots, m$$

Objective: estimation of the time dependent *volatility matrix*:

$$\Sigma^{jk}(t) = \sum_{i=1}^{d} \sigma_i^j(t)\sigma_i^k(t) \qquad j, k = 1, \ldots, n$$

Main Issues

$p^*(t)$ asset log-price Brownian semimartingale $\Rightarrow$ integrated volatility/covariance

$$\int_0^t \Sigma^{ik}(s)ds = P-\lim_{n\to\infty} \sum_{0\le j<t2^n} \left(p^i((j+1)2^{-n}) - p^i(j2^{-n})\right)\left(p^k((j+1)2^{-n}) - p^k(j2^{-n})\right).$$

Nevertheless, when sampling high frequency returns, three **difficulties** arise:

1) the distortion from efficient prices due to the **market microstructure** noise such as price discreteness, infrequent trading,...[Roll, 1984].
2) instantaneous volatility computation involves a sort of **numerical derivative**, which gives rise to numerical instabilities [Foster and Nelson, 1996, Comte and Renault, 1998]

In the multivariate case also:

3) the **non-synchronicity** of the arrival times of trades across markets leads to a bias towards zero in correlations among stocks as the sampling frequency increases [Epps, 1979]

Mean covariance [Malliavin and M. 2002, 2009]

Theorem

Consider a process p satisfying the assumption **(B)**. Then we have:

$$\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij}) = \mathcal{F}(dp^i) *_B \mathcal{F}(dp^j). \tag{1}$$

The convergence of the convolution product (1) is attained in probability

where, for $k \in \mathbf{Z}$

$$\mathcal{F}(dp^i)(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\, dp^i(t)$$

$$(\Phi *_B \Psi)(k) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{s=-N}^{N} \Phi(s)\Psi(k-s)$$

$$\mathcal{F}(\Sigma^{ij})(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\, \Sigma^{ij}(t)\, dt$$

Fourier instantaneous covariance computation

By the theorem we gather all the Fourier coefficients of the volatility matrix by means of the Fourier transform of the log-returns. Then reconstruct the **co-volatility functions** $\Sigma^{ij}(t)$ from its Fourier coefficients by the Fourier-Fejer summation:
let for $i, j = 1, 2$ and for any $|k| \leq N$,

$$c_N^{ij}(k) := \frac{1}{2N+1} \sum_{|s| \leq N} \mathcal{F}(dp^i)(s)\mathcal{F}(dp^i)(k-s),$$

then

$$\Sigma^{ij}(t) = \lim_{N \to \infty} \sum_{|k| < N} (1 - \frac{|k|}{N}) c_N^{ij}(k) e^{\mathrm{i}kt}$$

Consistency

Given observation times $(t_i^1)_{0\leq i\leq n_1}$ and $(t_j^2)_{0\leq j\leq n_2}$, $\rho(n) := \rho^1(n_1) \vee \rho^2(n_2)$ and $\rho^*(n_*) = \max_{t_l^*} |t_{l+1}^* - t_l^*|$, define:

$$c_k(dp_{n_1}^1) := \frac{1}{2\pi}\sum_{i=0}^{n_1-1} e^{-\mathrm{i}kt_i^1}(p^1(t_{i+1}^1) - p^1(t_i^1))$$

$$c_k(dp_{n_2}^2) := \frac{1}{2\pi}\sum_{j=0}^{n_2-1} e^{-\mathrm{i}kt_j^2}(p^2(t_{j+1}^2) - p^2(t_j^2))$$

$$c_k(\Sigma^{12}) := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\Sigma^{12}(t)dt$$

Consistency

Define for any $|k| \leq N$

$$\alpha_k(N, p^1_{n_1}, p^2_{n_2}) = \frac{2\pi}{2N+1} \sum_{|s| \leq N} c_s(dp^1_{n_1}) c_{k-s}(dp^2_{n_2}). \tag{2}$$

Suppose that $N\rho(n) \to 0$ as $N, n \to \infty$. Then, for any k, in probability

$$\alpha_k(N, p^1_{n_1}, p^2_{n_2}) \to c_k(\Sigma^{12})$$

HP: continuity. In probability, uniformly in t,

$$\widehat{\Sigma}^{12}_{n_1,n_2,N}(t) := \sum_{|k| \leq N} (1 - \frac{|k|}{N}) \alpha_k(N, p^1_{n_1}, p^2_{n_2}) e^{ikt} \to \Sigma^{12}(t) \tag{3}$$

Asymptotic Normality

Suppose and $\rho(n)N^{4/3} \to 0$, $\rho(n)N^{2\alpha} \to \infty$, if $\alpha > \frac{2}{3}$ and assumption **(A)** holds. Then for any function $g \in Lip(\alpha)$, with compact support in $(0, 2\pi)$,

$$(\rho(n))^{-\frac{1}{2}} \int_0^{2\pi} g(t)(\widehat{\Sigma}^{12}_{n,N}(t) - \Sigma^{12}(t))dt$$

converges in law to a mixture of Gaussian distribution with variance

$$\int_0^{2\pi} H'(t)g^2(t)(\Sigma^{11}(t)\Sigma^{22}(t) + (\Sigma^{12}(t))^2)dt.$$

(A) $H(t)$ quadratic variation of time

(i) $\rho(n) \to 0$ and $n_i\rho(n) = 0(1)$ for $i = 1, 2$

(ii) $H_n(t) := \frac{n}{2\pi} \sum_{t^1_{i+1} \wedge t^2_{j+1} \leq t} (t^1_{i+1} \wedge t^2_{j+1} - t^1_i \vee t^2_j)^2 I_{\{t^1_i \vee t^2_j < t^1_{i+1} \wedge t^2_{j+1}\}} \to H(t)$ as $n \to \infty$

(iii) $H(t)$ is continuously differentiable

If data are synchronous and equally spaced then $H'(t) = 1$, [Mykland and Zhang, 2006]

Spot volatility estimators

Alternative estimators of **spot volatility**, NOT involving numerical derivative of realized volatility estimators:

[Genon-Catalot, Laredo and Picard, 1992]
[Fan and Wang, 2008]
[Hoffman, Munk and Schmidt-Hieber, 2010]
[Muller, Sen and Stadtmuller, 2011]
[Mancini, Mattiussi and Reno, 2012]

Model with microstructure

Microstructure effects

market microstructure effects (discreteness of prices, bid/ask bounce, etc.) cause the discrepancy between asset pricing theory based on semi-martingales and the data at very fine intervals

Model for the observed log-returns [M. and Sanfelici, *J.F. Econometrics, 2011*]

$$\widetilde{p}^i(t) := p^i(t) + \eta^i(t) \quad \text{for } i = 1, 2,$$

Assumptions:

(M)

M1. $p := (p^1, p^2)$ and $\eta := (\eta^1, \eta^2)$ are independent processes, moreover $\eta(t)$ and $\eta(s)$ are independent for $s \neq t$ and $E[\eta(t)] = 0$ for any t.
M2. $E[\eta^i(t)\eta^j(t)] = \omega_{ij} < \infty$ for any t, $i, j = 1, 2$.

or **(MD)**

the microstructure noise is correlated with the price process and there is also a temporal dependence in the noise components

Fourier estimator of integrated covariance

$$\widehat{\Sigma}^{12}_{N,n_1,n_2} := \frac{(2\pi)^2}{2N+1} \sum_{|s|\le N} c_s(dp^1_{n_1}) c_{-s}(dp^2_{n_2})$$

If $\rho(n)N \to 0$, the following convergence in probability holds:

$$\lim_{n_1,n_2,N\to\infty} \widehat{\Sigma}^{12}_{N,n_1,n_2} = \int_0^{2\pi} \Sigma^{12}(t)dt.$$

In the application we consider also the following version which preserves **definite positiveness of the covariance matrix**

$$\widehat{\Sigma}^{12}_{N,n_1,n_2} := \frac{(2\pi)^2}{N+1} \sum_{|s|\le N} (1 - \frac{|s|}{N}) c_s(dp^1_{n_1}) c_{-s}(dp^2_{n_2}).$$

Quadratic covariation type estimators

Estimators based on the choice of a synchronization procedure, which gives the observations times $\{0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq 2\pi\}$ for both assets

Realized covariation $$RC^{12} := \sum_{i=1}^{n-1} \delta_i(p^1)\delta_i(p^2),$$

Realized covariation with leads and lags $$RCLL^{12} := \sum_{i} \sum_{h=-l}^{L} \delta_{i+h}(p^1)\delta_i(p^2),$$

Realized covariance kernels estimator $$RCLLW^{12} := \sum_{i} \sum_{h=-l}^{L} w(h)\delta_{i+h}(p^1)\delta_i(p^2),$$

where $\delta_i(p^*) = p^*(\tau_{i+1}) - p^*(\tau_i)$, and $w(h)$ is a kernel.

inconsistent for asynchronous observations and **inconsistent under (i.i.d) noise**, the MSE diverges as the number of observations increases; $RCLL^{1,2}$, $RCLLW^{1,2}$ more robust to microstructure noise, but they are much biased by dependent noise contaminations [Griffin and Oomen, 2010]

Refresh times consistent estimators

• [Barndorff-Nielsen, Hansen, Lunde and Shephard, 2008a] Realized covariance kernels with refresh times consistent for asynchronous observations/robust to some kind of noise

$$K^{12} := \sum_{h=-n}^{n} k\left(\frac{h}{H+1}\right)\Gamma_h^{12},$$

Γ_h^{12} is h-th realised autocovariance of the two assets, $k(\cdot)$ belongs to a suitable class of kernel functions (Parzen).

refresh time: choose the first time when both posted prices are updated, setting the price of the quicker asset to its most recent value (last-tick interpolation)

• [Kinnebrock and Podolskij, 2008] Modulated Realised Covariation

pre-averaging technique to reduce the microstructure effects (if one averages a number of observed log-prices, one is closer to the latent process $p(t)$)

Consistent estimators

- [Hayashi and Yoshida, 2005] All-overlapping estimator

$$AO^{12} := \sum_{i,j} \delta_{I_i^1}(p^1)\delta_{I_j^2}(p^2) I_{(I_i^1 \cap I_j^2 \neq \emptyset)},$$

where $\delta_{I_i^*}(p^*) := p^*(t_{i+1}^*) - p^*(t_i^*)$. Consistent for asynchronous observations, but NOT robust to noise: $\Rrightarrow$

- [Voev et Lunde, 2007] Sub-sampled All-overlapping estimator
- [Christensen, Podolskij and Vetter, 2012] Pre-averaged All-overlapping estimator

MSE

regular asynchronous trading: the asset 1 trades at regular points: $\Pi^1 = \{t_i^1 : i = 1, \ldots, n_1$ and $t_{i+1}^1 - t_i^1 = \frac{2\pi}{n_1}\}$; also asset 2 trades at regular points: $\Pi^2 = \{t_j^2 : j = 1, \ldots, n_2$ and $t_{j+1}^2 - t_j^2 = \frac{4\pi}{n_1}\}$, but no trade of asset 1 occurs at the same time of a trade of asset 2

$$MSE_{AOm} = o(1) + 2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt] + 2\omega_{22} \sum_{i=1}^{n-1} E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{11}(t)dt] + $$
$$+2(n-1)\omega_{11}\omega_{22}$$

$$MSE_{Fm} = o(1) + 2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} D_N^2(t_{n-1}^1 - t_j^2) E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt] +$$
$$+2\omega_{22} \sum_{i=1}^{n-1} D_N^2(t_i^1 - t_{\frac{n}{2}-1}^2) E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{11}(t)dt] + 4\omega_{11}\omega_{22} D_N^2(t_{n-1}^1 - t_{\frac{n}{2}-1}^2)$$

where $D_N(t) := \frac{1}{2N+1} \frac{\sin[(N+\frac{1}{2})t]}{\sin \frac{t}{2}}$

Optimal MSE-based Fourier estimator

These estimates allow to measure the MSE of the co-volatility estimators also in the case of empirical market quote data. Therefore, they can be used to build **optimal MSE-based estimators** by choosing the cutting frequency N which minimizes the estimated MSE instead of the true one.

Montecarlo Analysis

We simulate discrete data from the continuous time bivariate GARCH model

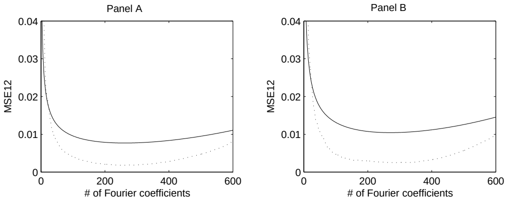
$$\begin{bmatrix} dp^1(t) \\ dp^2(t) \end{bmatrix} = \begin{bmatrix} \beta_1\sigma_1^2(t) \\ \beta_2\sigma_4^2(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \\ \sigma_3(t) & \sigma_4(t) \end{bmatrix} \begin{bmatrix} dW_5(t) \\ dW_6(t) \end{bmatrix}$$

$$d\sigma_i^2(t) = (\omega_i - \theta_i\sigma_i^2(t))dt + \alpha_i\sigma_i^2(t)dW_i(t), \quad i = 1, \dots, 4,$$

The logarithmic noises $\eta^1(t), \eta^2(t)$ are i.i.d. Gaussian, possibly contemporaneously correlated.

We generate second-by-second return and variance paths over a daily trading period of $h = 6$ hours. Then we sample the observations according to different scenarios: *regular synchronous trading* with durations $\rho_1 = \rho(n_1)$ and $\rho_2 = 2\rho_1$; *regular non-synchronous trading* with durations ρ_1 and $\rho_2 = 2\rho_1$ and displacement $\delta \cdot \rho_1$; *Poisson trading* with durations between trades drawn from an exponential distribution with means λ_1, λ_2.

Real (:) and estimated (-) MSE for $\hat{\Sigma}^{12}_{N,n_1,n_2}$ as a function of the cutting frequency N_{cut}. Panel A: regular non-synchronous trading setting, with $\rho_1 = 5$ sec, $\rho_2 = 10$ sec, $\delta = 2/3$ and uncorrelated i.i.d. noise. Panel B: regular non-synchronous trading setting, with $\rho_1 = 5$ sec, $\rho_2 = 10$ sec, $\delta = 2/3$ and correlated i.i.d. noise. Estimated MSE provides an upper bound of the actual one, can be used to find out an optimal cutting frequency N_{cut}



	Reg-NS		Reg-S + Unc		Reg-NS + Unc		Reg-NS + Cor	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	5.72e-4	-9.88e-3	3.35e-4	-6.09e-3	7.29e-4	-1.12e-2	4.73e-4	-8.82e-3
$RC^{12}_{0,5min}$	2.96e-2	-1.68e-1	1.06e-3	8.80e-4	3.45e-2	-1.80e-1	3.20e-2	-1.74e-1
RC^{12}_{1min}	9.14e-3	-8.44e-2	2.08e-3	2.70e-3	1.12e-2	-9.16e-2	9.74e-3	-8.65e-2
RC^{12}_{5min}	1.16e-2	-1.80e-2	1.14e-2	5.00e-3	1.44e-2	-2.33e-2	1.13e-2	-1.68e-2
$RCLL^{12}_{0,5min}$	2.88e-3	-1.68e-3	3.34e-3	2.94e-3	3.71e-3	-2.43e-3	3.15e-3	-1.55e-3
$RCLL^{12}_{1min}$	6.40e-3	-3.13e-3	6.42e-3	5.04e-3	8.00e-3	-3.37e-4	6.13e-3	3.09e-3
$RCLL^{12}_{5min}$	3.35e-2	1.11e-2	3.12e-2	3.15e-4	4.23e-2	-7.22e-3	3.61e-2	6.79e-3
AO^{12}	4.72e-4	-1.20e-3	4.47e-4	-1.08e-3	6.88e-4	9.45e-4	5.98e-4	-5.91e-4
K^{12}	9.33e-4	-8.13e-3	9.13e-4	-5.22e-4	1.28e-3	-6.32e-3	1.09e-3	-7.18e-3
MRC^{12}	2.80e-3	-3.27e-2	2.57e-3	-2.55e-2	3.38e-3	-3.01e-2	2.91e-3	-2.87e-2

	Reg-NS + Dep		Poisson + Unc		Poisson + Cor		Poisson + Dep	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	3.96e-4	-6.32e-3	1.07e-3	-1.38e-2	1.18e-3	-1.53e-2	1.00e-3	-1.43e-2
$RC^{12}_{0,5min}$	3.02e-2	-1.66e-1	3.33e-2	-1.76e-1	3.11e-2	-1.70e-1	2.91e-2	-1.64e-1
RC^{12}_{1min}	9.97e-3	-8.17e-2	1.08e-2	-8.95e-2	1.05e-2	-8.85e-2	1.03e-2	-8.62e-2
RC^{12}_{5min}	1.47e-2	-1.70e-2	1.28e-2	-2.50e-2	1.36e-2	-2.06e-2	1.23e-2	-2.64e-2
$RCLL^{12}_{0,5min}$	4.42e-3	3.20e-3	3.81e-3	-7.98e-3	3.40e-3	-6.84e-3	3.73e-3	-9.08e-3
$RCLL^{12}_{1min}$	8.06e-3	-9.21e-4	6.81e-3	-3.41e-3	7.23e-3	1.26e-3	7.80e-3	3.78e-3
$RCLL^{12}_{5min}$	3.59e-2	-1.60e-2	3.31e-2	-3.59e-3	3.74e-2	6.35e-3	3.67e-2	-1.47e-2
AO^{12}	7.42e-3	7.46e-2	1.29e-3	-8.75e-4	1.24e-3	9.32e-3	8.10e-3	7.49e-2
K^{12}	5.25e-3	5.43e-2	5.88e-3	-6.35e-2	4.57e-3	-5.46e-2	2.85e-3	-1.95e-2
MRC^{12}	3.93e-3	-1.59e-2	4.19e-3	-3.00e-2	3.71e-3	-2.71e-2	4.72e-3	-2.24e-2

Tabella: Comparison of integrated volatility estimators. The noise variance is 90% of the total variance for 1 second returns. $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

	Reg-S + Unc		Reg-NS + Unc		Reg-NS + Cor		Reg-NS + Dep	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	3.42e-4	-3.28e-3	3.93e-4	-4.93e-3	4.37e-4	-3.86e-3	8.67e-4	-4.90e-3
$RC^{12}_{0,5min}$	3.81e-2	4.01e-3	6.92e-2	-1.66e-1	8.73e-2	-1.81e-1	2.00e+0	-1.47e-1
RC^{12}_{1min}	2.26e-2	-4.08e-3	3.35e-2	-8.09e-2	4.31e-2	-8.67e-2	1.14e+0	-1.19e-1
RC^{12}_{5min}	1.93e-2	-4.05e-3	2.21e-2	-1.48e-2	2.67e-2	-8.87e-3	2.84e-1	-5.89e-2
$RCLL^{12}_{0,5min}$	2.77e-2	5.92e-3	3.46e-2	-1.57e-3	4.28e-2	2.48e-3	1.37e+0	-3.36e-2
$RCLL^{12}_{1min}$	2.29e-2	-1.27e-3	2.59e-2	-9.86e-4	3.45e-2	-8.57e-3	6.82e-1	1.37e-2
$RCLL^{12}_{5min}$	4.47e-2	1.02e-3	4.46e-2	1.02e-3	4.91e-2	1.48e-2	2.22e-1	-6.84e-4
AO^{12}	9.76e-2	5.38e-3	7.71e-2	2.49e-2	9.23e-2	-7.94e-3	4.40e+0	-8.95e-3
K^{12}	3.69e-2	-2.57e-3	3.80e-2	1.67e-2	4.94e-2	-7.48e-3	2.14e+0	2.44e-2
MRC^{12}	6.42e-3	-1.66e-2	7.74e-3	-1.40e-2	8.04e-3	-9.84e-3	1.25e-2	-2.21e-2

	Poisson + Unc		Poisson + Cor		Poisson + Dep	
	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	1.14e-3	-1.26e-2	5.35e-4	-5.62e-3	5.24e-4	-3.54e-3
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RC^{12}_{1min}	4.71e-2	-1.04e-1	3.00e-2	-1.54e-2	1.03e+0	-6.62e-2
RC^{12}_{5min}	2.79e-2	-3.07e-2	2.39e-2	-1.75e-2	3.01e-1	-3.93e-2
$RCLL^{12}_{0,5min}$	4.13e-2	-1.00e-2	3.70e-2	3.25e-4	1.43e+0	6.61e-2
$RCLL^{12}_{1min}$	3.18e-2	1.08e-2	2.87e-2	-8.09e-3	6.96e-1	-3.81e-2
$RCLL^{12}_{5min}$	5.88e-2	1.61e-2	4.39e-2	-2.27.e-3	2.40e-1	-3.03e-2
AO^{12}	8.83e-2	5.85e-3	1.27e+0	1.07e+0	2.91e+0	1.12e-1
K^{12}	4.87e-2	-5.59e-2	2.63e-1	4.70e-1	1.61e+0	1.83e-3
MRC^{12}	1.23e-2	-2.12e-2	9.94e-3	-2.22e-2	1.58e-2	-2.66e-2

Tabella: Comparison of integrated volatility estimators. Increased Noise (as in [Griffin and Oomen, 2010]). $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

Feasible estimators

In order to produce **feasible central limit theorems** for all the estimators, and as a consequence feasible confidence intervals, it is necessary to obtain **efficient estimators of the so called quarticity**, which appears as conditional variance of asymptotic distribution of the error in the central limit theorems.

Nevertheless, the studies about estimation of quarticity are still few:

> estimating integrated quarticity reasonably efficiently is a tougher problem than estimating the integrated volatility, as the effect of noise is magnified up

[Barndorff-Nielsen, Hansen, Lunde and Shephard, 2008a]

Fourier Quarticity estimator

- **First Step**: Computation of the Fourier coefficients of the volatility [Malliavin and M. 2002 *Fin. Stoch.*, 2009 *Ann. Stat.*]
- **Second step**: [M. and Sanfelici, *Quant. Finance*] Computation of the k-th Fourier coefficient of $\sigma^4(t)$, by the formula of Fourier series of a product.

Theorem

Under the assumption **(B)**, *the following convergence in probability holds*

$$\mathcal{F}(\sigma^4)(k) = \lim_{M\to\infty} \sum_{|s|\le M} \mathcal{F}(\sigma^2)(s)\,\mathcal{F}(\sigma^2)(k-s) \tag{4}$$

$$\int_0^{2\pi} \sigma^4(t)dt = 2\pi\mathcal{F}(\sigma^4)(0).$$

Note: in order to compute the integrated fourth power of volatility function the knowledge of the integrated volatility is not sufficient, but (all) the Fourier coefficients of the volatility are needed.

Spot quarticity

The fourth power of volatility function can be reconstructed by means of its Fourier coefficients (4) as the following limit in probability

$$\sigma^4(t) = \lim_{N\to\infty} \sum_{|k|<N} (1 - \frac{|k|}{N})\mathcal{F}(\sigma^4)(k)\exp(\mathrm{i}kt) \quad \text{for all } t \in (0, 2\pi)$$

Fourier quarticity estimator estimator

Define the **Fourier estimator of quarticity** by

$$\boxed{\sigma^4_{n,N,M} := 2\pi \ \sum_{|s|<M}(1 - \frac{|s|}{M})c_s(\sigma^2_{n,N})c_{-s}(\sigma^2_{n,N})}$$

We have chosen the Fourier-Fejer summation, which improves the behavior of the estimator for very high observation frequencies.

Effectiveness of Fourier estimation method when applied to compute the quarticity in the presence of microstructure noise, due to the intrinsic robustness of the Fourier estimator of volatility

Consistency of Fourier quarticity estimator

Theorem

If $\rho(n)NM \to 0$ *and* $\frac{M^2}{N} \to 0$ *as* $M, N, n \to \infty$, *then the following convergence in probability holds*

$$\lim_{n,N,M\to\infty} \sigma^4_{n,N,M} = \int_0^{2\pi} \sigma^4(t)dt$$

Optimal MSE-based Fourier estimator: This result establishes a link between the number of observations n and the parameters M, N. In order to obtain a feasible finite sample estimator of the integrated quarticity, we compute the analytical expression for the MSE of the Fourier quarticity estimator, thus providing a practical way to optimize the finite sample performance of the Fourier estimator as a function of the number of frequencies M and N by the minimization of the estimated mean squared error (MSE), for a given number of intra-daily observations n.

Consider the following model for the observed log-returns

$$\widetilde{p}(t) := p(t) + \eta(t)$$

(A.I) The random shocks $\eta(t_j)$, for any j, are independent and identically distributed with mean zero and bounded fourth moment.

(A.II) The true return process $\delta_j(p)$ is independent of $\eta(t_j)$ for any j.

Noise Bias

Under the assumptions **(B),(A.I),(A.II)**, then

$$Noise\ Bias = \Lambda_{n,N,M}(\sigma, \eta) + \Psi_{n,N,M}(\eta),$$

where $\Lambda_{n,N,M}(\sigma, \eta)$ goes to 0 under the conditions $\frac{MN^2}{n} \to 0$ and $\frac{M^3}{N} \to 0$, as $n, N, M \to \infty$, and

$$\Psi_{n,N,M}(\eta) = \frac{2}{\pi}(E[\eta^4] + 3E[\eta^2]^2) nMD_N^2(\frac{2\pi}{n}) \tag{5}$$

$D_N(t) := \frac{1}{2N+1} \frac{\sin[(N+\frac{1}{2})t]}{\sin \frac{t}{2}}$ denotes the rescaled Dirichlet kernel.

Corrected Fourier Estimator

- in order to obtain feasible optimal estimators we computed the analytical expression for the asymptotically vanishing term $\Lambda_{n,N,M}(\sigma, \eta)$
- a more efficient estimator of quarticity in the presence of noise can be constructed with the following correction

$$\boxed{\hat{\sigma}^4_{n,N,M} := \widetilde{\sigma}^4_{n,N,M} - \frac{M}{\pi} D_N^2\left(\frac{2\pi}{n}\right) \sum_{j=0}^{n-1} \delta_j(\tilde{p})^4}$$

where $\widetilde{\sigma}^4_{n,N,M}$ denotes Fourier quarticity estimator under noise observations

Monte Carlo simulation

We simulate second-by-second return and variance paths over a daily trading period of $T = 6$ hours, for a total of 252 trading days and $n = 21600$ observations per day.
CIR square-root model

$$\begin{aligned} dp(t) &= \sigma(t)\, dW_1(t) \\ d\sigma^2(t) &= \alpha(\beta - \sigma^2(t))dt + \nu\sigma(t)\, dW_2(t), \end{aligned} \tag{6}$$

W_1, W_2 independent Brownian motions

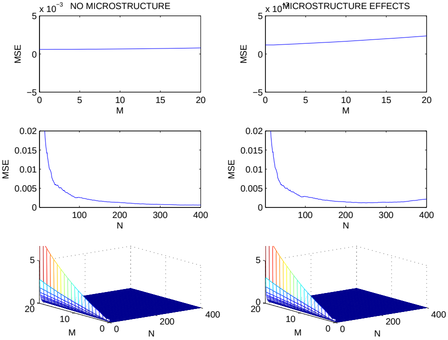
Parameters' values: $\alpha = 0.01$, $\beta = 1.0$, $\nu = 0.05$, $\sigma^2(0) = 1$ and $p(0) = \log 100$ (see Appendix [Bandi and Russell, 2005]). The logarithmic noises η are Gaussian i.i.d. and independent from p; we consider a noise-to-signal ratio of $\zeta = 2$ or $\zeta = 4$.

Choice of M and N

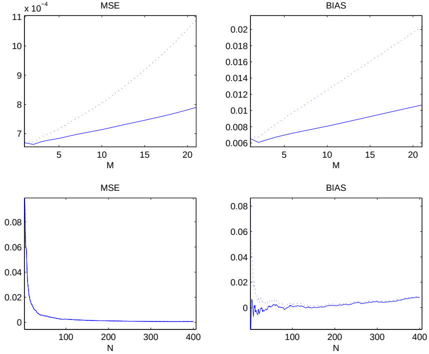
N is the most critical parameter in the design of the Fourier estimator, especially in the presence of noise, as

- the choice of N is crucial for an efficient computation of the volatility coefficients $c_s(\sigma^2_{n,N})$, which are the bricks used to build the quarticity estimate
- most of the microstructure is filtered out by truncating the volatility coefficients up to N, thus neglecting the noisy highest frequency return coefficients
- the MSE of the non corrected Fourier estimator tends to increase for large values of M and $N \Rightarrow$ need for the noise correction to further reduce the growth of the MSE with respect to M and for an accurate choice of N to filter out the microstructure effects

MSE of $\tilde{\sigma}^4_{n,N,M}$ averaged over the whole dataset (252 days) as a function of M and N, $\zeta = 4$.



Effect of the noise correction on the MSE and BIAS. The dotted line refers to $\tilde{\sigma}^4_{n,N,M}$, while the solid line to the corrected estimator $\hat{\sigma}^4_{n,N,M}$.



Comparison analysis

- Realized quarticity type estimators use lower frequency (5-15 minutes)

$$RQ := \frac{n}{3T} \sum_{i=0}^{n-1} \delta_i(p)^4 \quad \text{[Barndorff-Nielsen and Shephard, 2002]}$$

realized bipower quarticity [Barndorff-Nielsen and Shephard, 2004a]

$$BQ := \frac{n}{T} \sum_{i=1}^{n-1} |\delta_i(p)|^2 |\delta_{i-1}(p)|^2,$$

realized power and bipower quarticity [Barndorff-Nielsen and Shephard, 2004b]

$$Q := \frac{n}{2T} \left(\sum_{i=0}^{n-1} \delta_i(p)^4 - \sum_{i=1}^{n-1} |\delta_i(p)|^2 |\delta_{i-1}(p)|^2 \right),$$

realized tripower quarticity [Andersen, Bollerslev, Frederiksen and Nielsen, 2006]

$$TQ_1 := \mu_{4/3}^{-3} \frac{n^2}{(n-2)T} \sum_{i=2}^{n-1} |\delta_i(p)|^{4/3} |\delta_{i-1}(p)|^{4/3} |\delta_{i-2}(p)|^{4/3},$$

realized quadpower quarticity [Barndorff-Nielsen and Shephard, 2006]

$$QQ := \mu_1^{-4} \frac{n}{T} \sum_{i=3}^{n-1} |\delta_i(p)||\delta_{i-1}(p)||\delta_{i-2}(p)||\delta_{i-3}(p)|,$$

($\mu_p = E(|Z|^p)$, Z is a standard normally distributed r.v.)

Existing methods

- Estimators using all data:

  subsampled realized (bipower) quarticity estimators [Ghysels and Sinko, 2007]

$$RQ_{sub} := \frac{1}{S} \sum_{s=1}^{S} RQ^{(s)}$$

  the $RQ^{(s)}$'s are computed on different non overlapping subgrids using skip-S returns

  preaveraging method [Jacod, Li, Mykland, Podolskij and Vetter, 2009]

$$Q_{av} = \frac{1}{3\theta^2 \psi_2^2} \sum_{i=0}^{n-k_n+1} (\bar{p}_i^n)^4 - \frac{\rho(n)\psi_1}{\theta^4 \psi_2^2} \sum_{i=0}^{n-2k_n+1} (\bar{p}_i^n)^2 \sum_{j=i+k_n}^{i+2k_n-1} (\delta_j(p))^2 + \frac{\rho(n)\psi_1^2}{4\theta^4 \psi_2^2} \sum_{i=0}^{n-3} (\delta_i(p))^2 (\delta_{i+2}(p))^2,$$

  where the pre-averaged price process is

$$\bar{p}_i^n = \frac{1}{k_n} \left(\sum_{j=k_n/2}^{k_n-1} p_{i+j} - \sum_{j=0}^{k_n/2-1} p_{i+j} \right), \quad \theta = k_n \sqrt{\rho(n)},\ \psi_1 = 1,\ \psi_2 = 1/12.$$

Note: *nearest neighbor truncation* estimators [Andersen, Dobrev and Schaumburg, 2011] are specifically designed to cope with jumps but are less efficient than the multipower variation statistics in scenarios without jumps

Comparison analysis

	Unfeasible		Feasible	
	MSE	BIAS	MSE	BIAS
$\tilde{\sigma}^4_{n,N,M}$	6.71e-4	6.72e-3	7.21e-4	8.09e-3
$\hat{\sigma}^4_{n,N,M}$	6.65e-4	6.27e-3	7.01e-4	6.75e-3
RQ	4.48e-3	4.08e-2	5.44e-3	3.18e-2
BQ	4.71e-3	2.29e-2	5.67e-3	3.01e-2
Q	5.46e-3	3.96e-2	7.45e-3	3.26e-2
TQ_1	5.45e-3	2.36e-2	7.34e-3	3.75e-2
TQ_2	5.19e-3	2.03e-2	6.99e-3	3.44e-2
$TQ^{(k)}$	5.89e-3	3.89e-2	8.41e-3	3.90e-2
QQ	5.38e-3	2.07e-2	7.21e-3	3.34e-2
RQ_{sub}	3.16e-3	2.90e-2	3.17e-3	2.78e-2
BQ_{sub}	7.59e-4	-1.43e-2	2.41e-3	-9.56e-3
Q_{av}	3.39e-4	-6.81e-3	4.36e-4	-3.37e-3

Tabella: Microstructure effects ($\zeta = 2$). "Feasible": the estimators have been optimized with the rules provided by the literature for the other estimators, and with the feasible MSE minimization for Fourier estimator. "Unfeasible" stands for the "non feasible minimization of the real MSE". Optimal feasible sampling interval for realized type estimators is approx 2 min.

	Unfeasible		Feasible	
	MSE	BIAS	MSE	BIAS
$\tilde{\sigma}^4_{n,N,M}$	1.18e-3	1.62e-2	1.31e-3	1.71e-2
$\hat{\sigma}^4_{n,N,M}$	7.43e-4	1.14e-4	1.03e-3	-9.57e-4
RQ	1.05e-2	1.11e-2	1.38e-2	5.79e-2
BQ	1.21e-2	3.17e-2	1.68e-2	6.05e-2
Q	1.36e-2	3.23e-2	1.74e-2	5.66e-2
TQ_1	1.49e-2	3.14e-2	2.07e-2	6.77e-2
TQ_2	1.35e-2	2.25e-2	1.91e-2	6.04e-2
$TQ^{(k)}$	1.36e-2	3.81e-2	1.90e-2	6.64e-2
QQ	1.37e-2	1.52e-2	1.87e-2	5.51e-2
RQ_{sub}	7.08e-3	3.35e-2	7.35e-3	4.80e-2
BQ_{sub}	8.37e-4	-1.55e-2	4.96e-3	-1.48e-2
Q_{av}	5.05e-4	-3.59e-3	7.55e-4	-1.83e-3

Tabella: Microstructure effects ($\zeta = 4$). Same format as Table 3. Optimal feasible sampling interval for realized type estimators is approx 4 min.

	Unfeasible		Feasible	
	MSE	BIAS	MSE	BIAS
$\sigma^4_{n,N,M}$	6.62e-4	-2.60e-3	7.01e-4	3.52e-4
RQ	1.06e-3	9.03e-3	2.52e-3	-2.21e-3
BQ	8.69e-4	-1.25e-2	2.42e-3	-6.19e-3
Q	1.69e-3	5.47e-3	3.80e-3	-2.21e-4
TQ_1	1.12e-3	-1.63e-2	2.72e-3	-6.64e-3
TQ_2	1.13e-3	-1.69e-2	2.71e-3	-8.45e-3
$TQ^{(k)}$	1.02e-3	-1.05e-2	2.66e-3	-2.96e-3
QQ	1.32e-3	-1.94e-2	3.10e-3	-8.47e-3
RQ_{sub}	7.85e-4	1.03e-2	1.86e-3	-4.85e-3
BQ_{sub}	3.31e-3	-3.76e-2	3.81e-3	-4.83e-2
Q_{av}	7.93e-4	-5.95e-3	8.79e-4	-7.67e-3

Tabella: Irregular trading times and no noise. [Andersen, Dobrev and Schaumburg, 2011]: realized quarticity estimators are badly affected by irregular trading (they assume equal spacing and involve a multiplication by n/T). We simulate a scenario with Poisson irregular trading times with durations between observations drawn from an exponential distribution with means $\lambda = 5$ sec. Although no microstructure effects are taken into account, the optimal sampling interval for the realized quarticity-type estimators ranges from 0.4 to 0.69 min

	Unfeasible		Feasible	
	MSE	BIAS	MSE	BIAS
$\tilde{\sigma}^4_{n,N,M}$	1.79e-3	1.93e-2	4.87e-3	6.14e-2
$\hat{\sigma}^4_{n,N,M}$	9.36e-4	-1.31e-3	2.45e-3	3.72e-2
RQ	4.94e-3	3.96e-2	6.23e-3	3.06e-2
BQ	4.74e-3	3.72e-2	6.12e-3	2.43e-2
Q	6.68e-3	3.95e-2	8.66e-3	3.38e-2
TQ_1	5.42e-3	3.32e-2	7.10e-3	2.72e-2
TQ_2	5.16e-3	3.05e-2	6.75e-3	2.36e-2
$TQ^{(k)}$	5.50e-3	2.69e-2	8.39e-3	2.93e-2
QQ	5.40e-3	2.96e-2	7.24e-3	2.36e-2
RQ_{sub}	3.26e-3	2.41e-2	3.30e-3	2.19e-2
BQ_{sub}	2.88e-3	-2.81e-2	3.32e-3	-2.54e-2
Q_{av}	1.75e-3	1.43e-2	1.92e-3	2.46e-2

Tabella: Irregular trading times and microstructure effects ($\zeta = 2$). Same format as Table 3.

Multivariate case

- The Fourier method was originally proposed [Malliavin and M. 2002] for estimating multivariate volatility in order to overcome the difficulties arising by applying the quadratic covariation formula to the true return data, due to the non-synchronicity of observed prices for different assets.
  Thus we can extend without essential changes the univariate theory in order to obtain a high frequency estimator of the multivariate counterpart of quarticity.
  - First Step: Estimate the Fourier coefficients of the volatility matrix function
  - Second Step: Apply the product formula

[Barndorff-Nielsen and Shephard, 2004b] propose a consistent estimator of multivariate quarticity, but microstructure noise and asynchronicity is not considered.
[Christensen, Podolskij and Vetter, 2012] combine local averages and the AO estimator

Fourier estimator properties

1) uses all the available observations, no synchronization of the original data: it is based on the **integration of the time series of returns** rather than on its differentiation

2) it is designed specifically **for high frequency data**: by cutting the highest frequencies, it uses as much as possible of the sample path without being more sensitive to market frictions

Focus

3) it allows to reconstruct the volatility/covariance as a **stochastic function of time**: we can handle the volatility function as an observable variable

Stochastic Volatility Model

$$\begin{cases} dp(t) = \sigma(t)dW_0(t) + a(t)dt \\ dv(t) = \gamma(t)dZ(t) + b(t)dt \end{cases}$$

$v(t) := \sigma^2(t)$ is the variance process,
W_0 and Z correlated Brownian motions: $\eta(t)dt = dW_0(t) * dZ(t)$

Compute pathwise the diffusion coefficients $\sigma(t)$, $\gamma(t)$ and the covariance between the price and the instantaneous variance, $\varrho(t)$, given the observation of the asset price trajectory $p(t)$, $t \in [0, T]$

[Malliavin and M., 2002 C.R.A.S., Ser.I] [Barucci and M., 2010 IJTAF]

Method

Compute pathwise the diffusion coefficients $\sigma(t)$, $\gamma(t)$ and the covariance between the price and the instantaneous variance, $\varrho(t)$, given the observation of the asset price trajectory $p(t), \ t \in [0, T]$

1. compute the Fourier coefficients of the unobservable instantaneous variance process $v(t), \ t \in [0, T]$ in terms of the Fourier coefficients of $p(t) \Rrightarrow v(t)$ is reconstructed from its Fourier coefficients by the Fourier-Fejer summation method

2. the instantaneous variance $v(t)$ is handled as an observable variable $\Rrightarrow$ we iterate the procedure to compute the volatility of the variance process identifying the two components: volatility of variance ($\gamma(t)$) and asset price-variance covariance ($\varrho(t)$)

3. finally compute $\eta(t)$ by to the identity $\varrho(t) = \eta(t)\sigma(t)\gamma(t)$ with $\sigma(t)$ and $\gamma(t)$ a.s. positive

Volatility of Volatility

- Derive an estimator for Fourier coefficients $(c_k(\gamma^2))$ of $\gamma^2(t)$ given the observations of the variance process:
  By parts
  $$c_k(dv_{n,M}) = \mathrm{i}kc_k(v_{n,M}) + \frac{1}{2\pi}(v_{n,M}(2\pi) - v_{n,M}(0)),$$
  where $c_k(v_{n,M})$ were computed from dp
- Let

$$c_k(\gamma^2_{n,N,M}) := \frac{2\pi}{2N+1} \sum_{|j|\le N} c_j(dv_{n,M})c_{k-j}(dv_{n,M})$$

- If $\frac{N^4}{M} \to 0$ and $M^{\frac{5}{4}}\rho(n) \to 0$ for $n, N, M \to \infty$

$$P - \lim_{n,N,M\to\infty} c_k(\gamma^2_{n,N,M}) = c_k(\gamma^2)$$

Leverage

To compute the instantaneous covariance $\varrho(t)$ we exploit the **multivariate version of Fourier estimator**

- obtain a consistent estimator of the k-th Fourier coefficient of $\varrho(t)$ starting from the Fourier coefficients of the observed asset returns

$$c_k(\varrho_{n,N,M}) = \frac{2\pi}{2N+1} \sum_{|j| \le N} c_j(dp_n) c_{k-j}(dv_{n,M})$$

- If $\frac{N^2}{M} \to 0$ and $M\rho(n) \to 0$ for $n, N, M \to \infty$, then

$$P - \lim_{n,N,M \to \infty} c_k(\varrho_{n,N,M}) = c_k(\varrho)$$

(Preliminary) Montecarlo Analysis

Replicate numerical experiment by [Bollerslev and Zhou, 2002] who apply a **generalized moment method (GMM)** exploiting high frequency data, to estimate ξ, $\xi\eta(=\varrho)$ and square root process:

$$dp(t) = \sqrt{v(t)}dW_0(t)$$

$$dv(t) = k(\theta - v(t))dt + \xi\sqrt{v(t)}dZ(t)$$

k=mean reversion
θ=long run
ξ= volatility of variance

W_0, Z are standard Brownian motions $dW_0(t) * dZ(t) = \eta dt$

Montecarlo Analysis

We consider three parameter scenarios suggested in [Bollerslev and Zhou, 2002]:

$$\text{Scenario A}: \quad \mathrm{k} = 0.03,\ \theta = 0.25,\ \xi = 0.1,$$

$$\text{Scenario B}: \quad \mathrm{k} = 0.1,\ \theta = 0.25,\ \xi = 0.1,$$

$$\text{Scenario C}: \quad \mathrm{k} = 0.1,\ \theta = 0.25,\ \xi = 0.2,$$

Two values of η: $\eta = -0.2$ and $\eta = -0.7$

True values	Mean		Median		Standard Deviation	
	T=1000	T=4000	T=1000	T=4000	T=1000	T=4000
Panel A						
$\xi\eta = -0.02$	-0.0220	-0.0221	-0.0125	-0.0262	0.2157	0.1474
$\xi = 0.1$	0.1040	0.1014	0.1040	0.1014	0.0890	0.0768
Panel A						
$\xi\eta = -0.07$	-0.0706	-0.0729	-0.0622	-0.0730	0.2201	0.2106
$\xi = 0.1$	0.1075	0.1048	0.1075	0.1048	0.0856	0.0138
Panel B						
$\xi\eta = -0.02$	-0.0181	-0.0282	-0.0177	-0.0201	0.2865	0.2488
$\xi = 0.1$	0.1012	0.1069	0.1012	0.1069	0.0699	0.0695
Panel B						
$\xi\eta = -0.07$	-0.0717	-0.0737	-0.1314	-0.0711	0.2828	0.2560
$\xi = 0.1$	0.1330	0.1075	0.1331	0.1075	0.1188	0.0753
Panel C						
$\xi\eta = -0.04$	-0.0469	-0.0409	-0.1394	-0.0373	0.2707	0.1987
$\xi = 0.2$	0.2023	0.2066	0.2341	0.2165	0.1474	0.0892
Panel C						
$\xi\eta = -0.14$	-0.1263	-0.1569	-0.1442	-0.1561	0.3380	0.0616
$\xi = 0.2$	0.1994	0.2006	0.1984	0.2130	0.1571	0.0926

Tabella: Average value, median value and standard deviation of ξ and of $\xi\eta$ for three parameter scenarios, two correlation values and two choices of the size of the simulation sample.

Simulation results are satisfactory. The mean and the median of the parameters obtained in Table 7 are similar to those obtained in [Bollerslev and Zhou, 2002], only the standard deviation is slightly higher.

Note: the methodology in [Bollerslev and Zhou, 2002] exploits the knowledge of the square root model that generates the asset price observations, our methodology instead is model free and is able to recover the parameters of the data generating process without making a parametric assumption.

The performance of Fourier method is comparable to the one of the parametric method proposed in [Bollerslev and Zhou, 2002].
This exercise is only an illustrative example to show the efficiency of the method: as a matter of fact, parametric methods exploiting the assumption of a model, are expected to outperform non parametric methods.
Further analysis on going, where microstructure contamination is included.

[Bandi and Renó, 2012]

Conclusion

We have seen that the Fourier estimator of covariance is:
(i) consistent under asynchronous trading,
(ii) positive definite,
(iii) efficient in the presence of various types of microstructure noise: asymptotically unbiased and the MSE of the Fourier estimator converges to a constant, as the number of observations increases,
(iv) further it allows us to treat volatility as an observable variable, thus we can exploit the knowledge of its path
$\Rightarrow$ a very interesting alternative especially when microstructure effects are particularly relevant in the available data

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