On intra-day option pricing

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Outline

Outline

The Uncoupled CTRW: Definition

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \; X(0) = 0. \tag{1}$$

$X(t)$ is a semi-Markov pure-jump process.

- $N(t) = \max\{n : \; T_n \leq t\}$ is a renewal counting process, each new count occurs at a random renewal epoch T_n;
- $Y_i = X(T_i) - X(T_{i-1})$ are i.i.d. random variables (random jumps);
- $J_i = T_i - T_{i-1}$ are i.i.d. random variables (random durations or sojourn times);
- Y_i and J_i are independent from each other for any i.

The Uncoupled CTRW: Plot

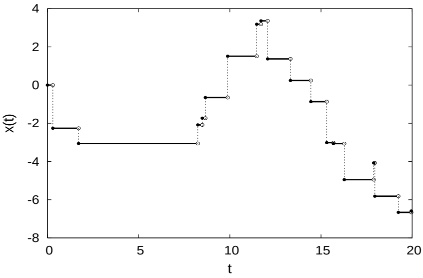


Figure: Realization of a CTRW with exponentially distributed waiting times ($\lambda = 1$) and standard normally distributed jumps ($\mu = 0$ and $\sigma = 1$).

The Uncoupled CTRW: Interpretation

- In Finance: $X(t) = \log[S(t)/S(0)]$, where $S(t)$ is the price of an asset at time t, T_n n-th trade time; J_i i-th intertrade duration; Y_i i-th tick-by-tick log-return.
- In Insurance: $X(t)$ sum of the claims paid up to time t, T_n n-th payment time; J_i i-th interpayment duration; Y_i i-th claim.
- In Economics (e.g. firm growth theory): $X(t) = \log[S(t)/S(0)]$, where $S(t)$ is the size of a firm at time t, T_n n-th growth event time; J_i i-th duration between shocks; Y_i i-th growth shock.
- In Physics: $X(t)$ position of a diffusing particle at time t; T_n time of n-th jump, J_i i-th sojourn time; Y_i i-th jump.

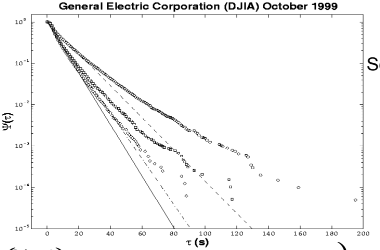
Outline

1. Motivation
   - The Uncoupled Continuous-Time Random Walk
   - Durations
2. Option pricing
   - Martingale option price
   - Ingredients
3. Conclusions
   - Conclusions
   - Acknowledgments
   - For Further Reading

Durations

Empirical results on the waiting-time survival function (Anderson-Darling test)

Interval 1 (9-11): 16063 data; τ_0 = 7 s
Interval 2 (11-14): 20214 data; τ_0 = 11.3 s
Interval 3 (14-17): 19372 data; τ_0 =7.9 s



See Engle and Russell (1994/1998)

$$A^2 = \left(-\sum_{i=1}^{n} \frac{(2i-1)}{n} \left[\ln \Psi(\tau_{n+1-i}) + \ln(1-\Psi(\tau_i)) \right] - n \right) \times (1 + (0.6/n))$$

where $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n$

$A_1^2 = 352$; $A_2^2 = 285$; $A_3^2 = 446 >> 1.957$ (1% significance)

Conclusion on durations

Durations are not exponentially distributed. This rules out the compound Poisson process, which is the only Markovian and Lévy pure-jump process. In particular, Merton's 1976 formula has to be generalized [1].

Outline

Equivalent martingale measure

One can replace Y_i in equation (1) with $\widetilde{Y}_i = Y_i - a$ defining a modified log-price process

$$\widetilde{X}(t) = \sum_{i=1}^{N(t)} \widetilde{Y}_i = \sum_{i=1}^{N(t)} (Y_i - a), \tag{2}$$

as well as the corresponding modified price process

$$\widetilde{S}(t) = \mathrm{e}^{\widetilde{X}(t)}. \tag{3}$$

Now, if $a = \log(\mathbb{E}(\mathrm{e}^{Y_i}))$, one has that $\widetilde{S}(t)$ is a martingale. In fact, one can write

$$\mathbb{E}(\widetilde{S}(t)|\mathcal{F}_s) = \widetilde{S}(s) \prod_{i=N(s)+1}^{N(t)} \mathbb{E}\left(\mathrm{e}^{Y_i - a}\right) = \widetilde{S}(s). \tag{4}$$

Martingale option price

Let $\widetilde{C}(S(T_M))$ represent the *pay-off* of a European call option at maturity. Then, the option price $C(t)$ at a time $t < T_M$ is given by the discounted conditional expected value of the pay-off at maturity with respect to the e.m.m., that is

$$C(t) = \mathrm{e}^{r(t-T_M)}\mathbb{E}_{\widetilde{\mathbb{S}}}(\widetilde{C}(S(T_M))|\mathcal{F}_t), \qquad (5)$$

where r is the risk-free interest rate. For intra-day data, it is safe to assume $r = 0$, so that equation (5) simplifies to

$$C(t) = \mathbb{E}_{\widetilde{\mathbb{S}}}(\widetilde{C}(S(T_M))|\mathcal{F}_t). \qquad (6)$$

Martingale option price continued

In the general case in which t is a generic observation time not coinciding with a renewal epoch, things become tricky, even if we are using a simplified and stylized model. At time t, both the price $S(t)$ and the number of trades $N(t) = n_t$ are known. We can consider the random variable $\Delta X(t, T_M) = X(T_M) - X(t) = \log(S(T_M)/S(t))$. If $S(t)$ is used as numeraire (that is if we set $S(t) = 1$), Equation (6) becomes

$$C(t) = \mathbb{E}_{\widetilde{\mathbb{S}}}(\widetilde{C}(S(T_M))|\mathcal{F}_t) = \int_0^\infty \widetilde{C}(u) dF^{n_t}_{\widetilde{S}(T_M)}(u), \tag{7}$$

where the cumulative distribution function $F^{n_t}_{\widetilde{S}(T_M)}(u)$ is given by

$$F^{n_t}_{\widetilde{S}(T_M)}(u) = \sum_{n=0}^{\infty} \mathbb{P}(N(T_M) - N(t) = n | N(t) = n_t) F^{\star \mathcal{M} n}_{\widetilde{Y}}(u). \tag{8}$$

Outline

Ingredients: The counting distribution

In equation (8), $F_{\widetilde{Y}}^{\star_{\mathcal{M}} n}(u)$ represents the n-fold Mellin convolution giving the cumulative distribution for the product of n copies of the i.i.d. r.v. $\mathrm{e}^{\widetilde{Y}}$.

The probability distribution $\mathbb{P}(N(T_M) - N(t) = n | N(t) = n_t)$ of having n trades between time t and time T_M, given that there were n_t trades up to time t can be computed by elementary probabilistic methods. As derived in [2], this is given by

$$\mathbb{P}(N(T_M) - N(t) = n | N(t) = n_t) = \int_0^{T_M - t} \mathbb{P}(N(T_M) - N(t+u) = n-1) dF_{\mathcal{J}_{t,n_t}}(u). \quad (9)$$

Ingredients continued: The counting distribution

One has

$$\mathbb{P}(N(T_M) - N(t+u) = n-1) = \int_0^{T_M-(t+u)} (1 - F_J(T_M - (t+u+v))dF_J^{\star(n-1)}(v), \quad (10)$$

where $F_J^{\star(n-1)(v)}$ is the $n-1$-fold convolution of $F_J(v)$. $F_{\mathcal{J}_{t,n_t}}(u) = \mathbb{P}(\mathcal{J}_{t,n_t} \leq u)$ is the cumulative distribution function of the *residual life-time* at time t conditioned on the fact that there were n_t trades up to time t which we denote by $\mathcal{J}_{t,n_t}$. The residual life time is the time interval from t to the next renewal epoch $T_{N(t)+1}$.

Ingredients continued: The residual life time

The cumulative distribution function $F_{\mathcal{J}_{t,n_t}}(u)$ can be found by direct elementary probabilistic tools without using Laplace-tranform methods. We can see that the event $\mathcal{J}_{t,n_t} \leq u$ can be described in term of a conditional event (see [2, 3])

$$\{\mathcal{J}_{t,n_t} \leq u\} = \{T_{n_t+1} - t \leq u | N(t) = n_t\}. \tag{11}$$

Equation (11) can be written in terms of epochs using $\{N(t) = n_t\} = \{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\}$ which leads to:

$$\{\mathcal{J}_{t,n_t} \leq u\} = \{T_{n_t+1} - t \leq u | \{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\}\}. \tag{12}$$

Ingredients continued: The residual life time

One can now use the definition of conditional probability and the indicator function method to compute $F_{\mathcal{J}_{t,n_t}}(u)$ directly. First of all, one can write

$$\begin{aligned} F_{\mathcal{J}_{t,n_t}}(u) &= \mathbb{P}(\mathcal{J}_{t,n_t} \leq u) = \mathbb{P}(T_{n_t+1} - t \leq u | \{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\}) \\ &= \frac{\mathbb{P}(\{T_{n_t+1} - t \leq u\} \cap \{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\})}{\mathbb{P}(\{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\})}, \end{aligned} \quad (13)$$

and the denominator is already given by equation (10), meaning that one has

$$\mathbb{P}(\{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\}) = \mathbb{P}(N(t) = n_t) = \int_0^t (1 - F_J(t-w)) dF_J^{\star n_t}(w). \quad (14)$$

Ingredients continued: The residual life time

In order to compute the numerator, one can use the following equality between events

$$\{T_{n_t+1} - t \leq u\} \cap \{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\} = \{T_{n_t} \leq t\} \cap \{t - T_{n_t} < J_{n_t+1} \leq t + u - T_{n_t}\}, \quad (15)$$

and obtain that

$$\mathbb{P}(\{T_{n_t+1} - t \leq u\} \cap \{T_{n_t} \leq t\} \cap \{T_{n_t+1} > t\}) = \mathbb{P}(\{T_{n_t} \leq t\} \cap \{t - T_{n_t} < J_{n_t+1} \leq t + u - T_{n_t}\}) =$$

$$\mathbb{E}\left(I_{\{T_{n_t} \leq t\}} I_{\{t - T_{n_t} < J_{n_t+1} \leq t+u-T_{n_t}\}}\right) = \int_0^t \int_{t-w}^{u+t-w} dF_{T_{n_t}}(w) dF_J(v) =$$

$$\int_0^t \int_{t-w}^{u+t-w} dF_J^{\star n_t}(w) dF_J(v) = \int_0^t (F_J(u+t-w) - F_J(t-w)) dF_J^{\star n_t}(w). \quad (16)$$

The end of the ingredients!

Combining equations (14) and (16), from equation (13) one finally gets

$$F_{\mathcal{J}_{t,n_t}}(u) = \frac{\int_0^t (F_J(u+t-w) - F_J(t-w)) dF_J^{\star n_t}(w)}{\int_0^t (1 - F_J(t-w)) dF_J^{\star n_t}(w)}. \qquad (17)$$

Equation (17) is the last ingredient needed to determine the option price in the general case (7). Finally, note that equation (7) yields Merton's formula (in the absence of diffusion) when $J \sim \exp(\lambda)$ and $Y \sim N(\mu, \sigma^2)$.

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Conclusions

- It is possible to derive a formula for intra-day option prices using the martingale method and assuming the underlying follows an uncoupled CTRW (a compound renewal process).
- Numerical work is under way to compare this formula with Merton's result.
- The model presented here is not yet realistic, it does not include heteroscedasticity as well as correlations in durations.
- A market may develop for intra-day option both for hedging and speculative purposes.

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   - For Further Reading

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Outline

For Further Reading I

R.C. Merton
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Full Characterization of the Fractional Poisson Process
Europhysics Letters, **96**, 20004–20009, 2011.

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A Parsimonious Model for Intraday European Option Pricing.
Economics Discussion Papers, No 2012-14, Kiel Institute for the World Economy, 2012.

Merton's formula

For the sake of simplicity, we focus on the European call. Let us assume that the derivative position is opened at a time t after the start of continuous trading with maturity at a time T_M before the end of continuous trading. If $J \sim \exp(\lambda)$ and $Y \sim N(\mu, \sigma)$ (compound Poisson process) and for a vanishing risk-free interest rate (which is a reasonable assumption for intra-day data), one has the following formula for the plain-vanilla option price $C(t)$

$$C(t) = \mathrm{e}^{\lambda(t-T_M)} \sum_{n=0}^{\infty} \frac{(\lambda(T_M - t))^n}{n!} C_n(S(0), K, \mu, \sigma^2), \qquad (18)$$

Merton's formula continued

where λ is the activity of the Poisson process for trades, K is the strike price, μ and σ^2 are, respectively, the expected value and the variance of the log-price jumps which are assumed to be normally distributed. One further has that

$$C_n(S(0), K, \mu, \sigma^2) = N(d_{1,n})S(0) - N(d_{2,n})K, \tag{19}$$

where

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} dv \mathrm{e}^{-v^2/2} \tag{20}$$

is the standard normal cumulative distribution function and, finally

$$d_{1,n} = \frac{\log(S(0)/K) + n(\mu + \sigma^2/2)}{\sqrt{n}\sigma}, \tag{21}$$

$$d_{2,n} = d_{1,n} - \sigma\sqrt{n}. \tag{22}$$

Durations