On portfolio optimization in markets with frictions

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Part I

Merton's problem

Definition

We consider a financial market consisting of a riskless asset B_t and a risky asset S_t with the following dynamics:

- $dB_t = rB_t$;
- $dS_t = S_t(\mu dt + \sigma dW_t)$.

Definition

Let θ_t be the number of shares owned by an agent at time t, X_t its total wealth and $\Pi_t = \frac{\theta_t S_t}{X_t}$ the proportion of wealth invested in the stock.

The self-financing condition is given by

$$dX_t = \frac{X_t - \theta S_t}{B_t} dB_t + \theta dS_t.$$

The problem

From now on we will assume $r = 0$.

Given a utility function $U(\cdot)$, we want to maximize the expected utility from terminal wealth on a certain time horizon T:

$$\sup_{\theta} E[U(X_T)].$$

The problem can be completely solved in the case of a utility function with constant Relative Risk Aversion, i.e. when $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for $\gamma > 0$, $\gamma \neq 1$ or when $U(x) = \log x$.

Solution

The best strategy is to keep the the proportion Π_t constant and equal to $\frac{\mu}{\gamma\sigma^2}$, where γ is the relative risk-aversion.

H-J-B approach

Definition

The value function of the utility maximization problem is defined by

$$v(t,x) = \sup_{\theta \in \Theta} E[U(X_T^{t,x})]$$

Dynamic Programming Principle

$$v(t,x) = \sup_{\theta \in \Theta_{[t,t+h]}} E^{t,x}[v(t+h, X_{t+h}^{\theta})]$$

H-J-B approach

If we assume regularity of $v(\cdot,\cdot)$ and $X_\cdot$, we get from Itô's formula

$$E^{t,x}[v(t+h, X^{\theta}_{t+h})] = v(t,x) + E\int_t^{t+h} \frac{\partial v}{\partial t}(s, X^{\theta}_s) + (\mathcal{L}^{\theta} v)(s, X^{\theta}_s)ds,$$

where $\mathcal{L}^{\theta}$ is the Kolmogoroff operator associated to the process $X^{\theta}_{\cdot}$.

Thus, we have

$$\frac{\partial v}{\partial t}(t,x) + (\mathcal{L}^{\theta} v)(t,x) \leq 0$$

and the equality is obtained for the optimal strategy.

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H-J-B approach

Hamilton-Jacobi-Bellman equation

$$\frac{\partial v}{\partial t}(t,x) + \sup_{\theta \in \Theta}\{(\mathcal{L}^{\theta} v)(t,x)\} = 0;$$
$$v(T,x) = U(x).$$

In Merton's problem, the dynamics of X_t are given by

$$dX_t = X_t \Pi_t (\mu dt + \sigma dW_t)$$

and, considering Π_t as the control process,

$$\mathcal{L}^{\pi} = x\pi\mu \frac{\partial}{\partial x} + \frac{1}{2} x^2 \pi^2 \sigma^2 \frac{\partial^2}{\partial x^2}.$$

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H-J-B approach

The optimal proportion is given by

$$\Pi_t = -\frac{\mu}{\sigma^2}\frac{v_x(t,X_t)}{X_t v_{xx}(t,X_t)},$$

where $v(\cdot,\cdot)$ solves

- $v_t(t,x) - \frac{1}{2}\frac{v_x^2(t,x)}{v_{xx}(t,x)}\frac{\mu^2}{\sigma^2} = 0;$
- $v(T,x) = U(x)$.

For $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, the solution is given by $v(t,x) = \exp\{\beta(T-t)\}U(x)$, $\beta = \frac{\mu^2(1-\gamma)}{2\sigma^2\gamma}$ and $\Pi_t = \frac{\mu}{\gamma\sigma^2}$.
For $U(x) = \log x$, the solution is given by $v(t,x) = \beta(T-t) + U(x)$, $\beta = \frac{\mu^2}{2\sigma^2}$ and $\Pi_t = \frac{\mu}{\sigma^2}$.

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Duality approach

Observation

$$\mathcal{C}(x) = \{X \leq x + \int_0^T \theta_s dS_s : \theta_\cdot \in \Theta\} = \{X : E^Q[X] \leq x, \ \forall Q \in \mathcal{M}\},$$

where $\mathcal{M}$ is the set of martingale measures.

Definition

Let $\tilde{U}(\cdot)$ be the Legendre-Fenchel transformation defined as

$$\tilde{U}(y) = \sup_{x>0}[U(x) - xy] = U(I(y)) - yI(y),$$

where $I = (U')^{-1}$.

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Duality approach

Observation

$$E[U(X)] \leq E[\tilde{U}(yZ)] + E[yZX] \leq E[\tilde{U}(yZ)] + xy,$$

where $X \in \mathcal{C}(x)$ and $Z = \frac{dQ}{dP}$ for $Q \in \mathcal{M}$.

The dual problem is given by $\inf_{y>0, Z \in \mathcal{M}} \{E[\tilde{U}(yZ)] + xy\}$. Let $\hat{y}$, $\hat{Z}$ be the minimizers and set $\hat{X} = I(\hat{y}\hat{Z})$.

Duality approach

Merton considers a complete market, thus $\mathcal{M} = \{Q\}$. The corresponding change of measure is given by

$$Z = \exp\left\{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T\right\}.$$

The optimal $\hat{y}$ is such that $E[ZI(\hat{y}Z)] = x$.

If $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, then $I(y) = y^{-\frac{1}{\gamma}}$.

$$\hat{X}_t = E^Q[(\hat{y}Z)^{-\frac{1}{\gamma}}|\mathcal{F}_t] = x\exp\left\{\frac{\mu}{\gamma\sigma}W_t^Q - \frac{1}{2}\frac{\mu^2}{(\gamma\sigma)^2}t\right\},$$

i.e.

$$d\hat{X}_t = \hat{X}_t\frac{\mu}{\gamma\sigma}dW_t^Q = \hat{X}_t\frac{\mu}{\gamma\sigma^2}\sigma dW_t^Q = \hat{X}_t\frac{\mu}{\gamma\sigma^2}\frac{dS_t}{S_t}.$$

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Myopic Utility

The mainstream literature assumes a frictionless market.

Suppose

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t).$$

Set $Z = \exp\{-\int_0^T \frac{\mu_t}{\sigma_t} dW_t - \frac{1}{2}\int_0^T \frac{\mu_t^2}{\sigma_t^2} dt\}$ and $\hat{y}$ such that $E[ZI(\hat{y}Z)] = x$.

Observation

$$E[U(X_T^\pi)] \leq E[U(I(\hat{y}Z)) - \hat{y}Z(I(\hat{y}Z) - X_T^\pi)] \leq E[U(I(\hat{y}Z))].$$

Myopic Utility

Logarithmic case

When $U(x) = \log x$, we have $I(y) = \frac{1}{y}$ and $\hat{y}(x) = \frac{1}{x}$.
$\Pi_t = \frac{\mu_t}{\sigma_t^2}$ is the optimal proportion. Indeed, $X_T^\pi = I(\hat{y}Z)$ and
$v(0,x) = \log x + E \int_0^T \frac{1}{2} \frac{\mu_s^2}{\sigma_s^2} ds$.

The logarithmic utility is called myopic utility.

Part II

Transaction costs

Transaction costs

- If we take into account transaction costs, it is impossible to keep the Merton proportion in stocks. This would imply infinite trading, thus infinite loss.
- Davis and Norman in 1990 prove the existence of a "no-trading region".

We consider proportional transaction costs. To this end, we assume the existence of a bid-ask spread, i.e. S_t is the ask price and $(1-\lambda)S_t$ is the bid price, for some $\lambda \in (0,1)$.

The problem

We suppose that an agent can invest in a bond $B_t = 1$ and in a stock with ask price

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

and bid price $(1-\lambda)S_t$.

Definition

A trading strategy is a (predictable) finite variation process (θ_t^0, θ_t) with $(\theta_0^0, \theta_0) = (x, 0)$

Let $\theta_t = \theta_t^{\uparrow} - \theta_t^{\downarrow}$, where $\theta_t^{\uparrow}$ and $\theta_t^{\downarrow}$ are two increasing processes, which do not grow at the same time. Then the self-financing condition becomes

$$d\theta_t^0 = (1-\lambda)S_t d\theta_t^{\downarrow} - S_t d\theta_t^{\uparrow}.$$

The problem

Definition

A self-financing strategy is called admissible if its liquidation wealth process

$$X_t^{\theta^0,\theta} = \theta_t^0 + \theta_t^+(1-\lambda)S_t - \theta_t^- S_t$$

is a.s. nonnegative.

Our objective is to maximize

$$E[\log X_T^{\theta^0,\theta}]$$

over all possible trading strategies (θ^0, θ).

Shadow price

Definition

A shadow price is a continuous semimartingale $\tilde{S}$ evolving within the bid-ask spread $[(1-\lambda)S_t, S_t]$ such that the log-optimal portfolio for the frictionless market with price $\tilde{S}$ exists, is of finite variation and θ_t only increases (resp. decreases) on $\{\tilde{S}_t = S_t\}$ (resp. $\{\tilde{S}_t = (1-\lambda)S_t\}$).

Modified problem

Given a shadow price $\tilde{S}$, we call an admissible strategy (ψ_t^0, ψ_t) optimal for the modified problem problem if it maximizes

$$E[\log \tilde{X}_T^{\theta_t^0, \theta_t}],$$

where $\tilde{X}_T^{\theta_t^0, \theta_t} = \theta_T^0 + \theta_T \tilde{S}_T$.

Shadow price

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Shadow price

Proposition

Let $\tilde{S}$ be a shadow price and let (θ_t^0, θ_t) be the optimal strategy for the frictionless market with stock $\tilde{S}$ and logarithmic utility. Then (θ_t^0, θ_t) is also optimal for the modified problem.

Strategy

- Find a shadow price;
- solve the problem using techniques for the frictionless case.

Shadow price

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Doubly reflected geometric Brownian Motion

We will define a process with values in $[1, \bar{s}]$, which has the same dynamics as S in $(1, \bar{s})$.

Define the stopping time (ρ_n), (σ_n) and the processes (m_t), (M_t)

- $\rho_0 = 0$;
- $\sigma_n = \inf\{t \geq \rho_{n-1} : \frac{S_t}{m_t} \geq \bar{s}\}$, where $m_t = \inf_{\rho_{n-1} \leq u \leq t} S_u$ on $[\rho_{n-1}, \sigma_n]$;
- $\rho_n = \inf\{t \geq \sigma_n : \frac{S_t}{M_t} \leq \frac{1}{\bar{s}}\}$, where $M_t = \sup_{\sigma_n \leq u \leq t} S_u$ on $[\sigma_n, \rho_n]$.

We can continuously extend the process m as $m_t := \frac{M_t}{\bar{s}}$ on $\cup_n [\sigma_n, \rho_n]$.
The process $(\frac{S_t}{m_t})$ is a doubly reflected geometric Brownian Motion for the interval $[1, \bar{s}]$.

Smooth pasting

Let $g : [1, \bar{s}] \to [1, (1-\lambda)\bar{s}]$ be a $\mathcal{C}^2$ function such that $g' > 0$ and

- $g(1) = g'(1) = 1$;
- $g(\bar{s}) = (1-\lambda)\bar{s}$ and $g'(\bar{s}) = 1 - \lambda$.

Proposition

Define $\tilde{S}_t = m_t g(\frac{S_t}{m_t})$. Then $\tilde{S}$ is an Itô process with dynamics

$$d\tilde{S}_t = g'\left(\frac{S_t}{m_t}\right) dS_t + \frac{1}{2m_t} g''\left(\frac{S_t}{m_t}\right) (dS_t)^2$$

and takes values in $[(1-\lambda)S, S]$.

Idea: Apply Itô's formula assuming m_t constant. The smooth pasting conditions ensure that the diffusion term of $\frac{\tilde{S}_t}{S_t}$ vanishes when $S_t = m_t$ or $S_t = \bar{s}m_t$.

The function g

Suppose $S_{t_0} = m_{t_0} = 1$ and $\Pi_{t_0} = \frac{\theta_{t_0} S_{t_0}}{\theta^0_{t_0} + \theta_{t_0} S_{t_0}} = \frac{1}{1+\theta^0_{t_0}/\theta_{t_0}}$.
Suppose the process S_t moves upwards until, at time t_1, S_{t_1} reaches the level $\bar{s}$, where $\Pi_{t_1} = \frac{1}{1+\theta^0_{t_0}/(\theta_{t_0}\bar{s})}$ touches the selling boundary of the no-trading region.
On the interval $[t_0, t_1]$, we have $\tilde{S}_t = g(S_t)$. Since

$$\frac{dg(S_t)}{g(S_t)} = \left(\frac{\mu g'(S_t) S_t + \frac{\sigma^2}{2} g''(S_t) S_t^2}{g(S_t)} \right) dt + \left(\frac{\sigma g'(S_t) S_t}{g(S_t)} \right) dW_t,$$

the corresponding log-optimal proportion is given by

$$\frac{g(S_t)(\mu g'(S_t) S_t + \frac{\sigma^2}{2} g''(S_t) S_t^2)}{\sigma^2 g'(S_t)^2 S_t^2}.$$

It has to equate $\tilde{\Pi}_t = \frac{\theta_t \tilde{S}_t}{\theta^0_t + \theta_t \tilde{S}_t} = \frac{g(S_t)}{c+g(S_t)}$, where $c = \theta^0_{t_0}/\theta_{t_0}$ on $[t_0, t_1]$.

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The function g

This leads to the following equation for g:

$$g''(s) = \frac{2g'(s)^2}{c + g(s)} - \frac{2\mu g'(s)}{\sigma^2 s}.$$

With the boundary conditions $g(1) = g'(1) = 1$, the solution is

$$g(s) = \frac{-cs + (2\pi - 1 + 2c\pi)s^{2\pi}}{s - (2 - 2\pi + c(2\pi - 1))s^{2\pi}},$$

where $\pi = \frac{\mu}{\sigma^2} \neq \frac{1}{2}$.
After imposing $g(\bar{s}) = (1 - \lambda)\bar{s}$ and $g'(\bar{s}) = 1 - \lambda$, we can determine $\bar{s}$ and c.

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The result

Assume $\pi = \frac{\mu}{\sigma^2} \notin \{\frac{1}{2}, 1\}$. Let c be such that

$$\left(\frac{c}{(2\pi - 1 + 2c\pi)(2 - 2\pi - c(2\pi - 1))}\right)^{\frac{1-\pi}{\pi-1/2}} - \frac{1}{1-\lambda}(2\pi - 1 + 2c\pi)^2 = 0.$$

Set

$$\bar{s} = \left(\frac{c}{(2\pi - 1 + 2c\pi)(2 - 2\pi - c(2\pi - 1))}\right)^{1/(2\pi-1)}$$

and

$$g(s) = \frac{-cs + (2\pi - 1 + 2c\pi)s^{2\pi}}{s - (2 - 2\pi + c(2\pi - 1))s^{2\pi}}.$$

The result

Then $\tilde{S}_t = m_t g(\frac{S_t}{m_t})$ is a shadow price with log-optimal trading strategy given by

$$\theta_t^0 = \theta_{\rho_{k-1}}^0 \left(\frac{m_t}{m_{\rho_{k-1}}}\right)^{\frac{1}{c+1}} \text{ on } \bigcup_k [\rho_{k-1}, \sigma_k];$$

$$\theta_t^0 = \theta_{\sigma_k}^0 \left(\frac{m_t}{m_{\sigma_k}}\right)^{\frac{(1-\lambda)\bar{s}}{c+(1-\lambda)\bar{s}}} \text{ on } \bigcup_k [\sigma_k, \rho_k]$$

and $\theta_t^0 = c m_t \theta_t$.

The fraction $\tilde{\Pi}_t = \frac{1}{1+c/g(\frac{S_t}{m_t})}$ is kept in the no-trading region $[\frac{1}{1+c}, \frac{1}{1+c/((1-\lambda)\bar{s})}]$ and so the fraction $\Pi_t = \frac{1}{1+c\frac{m_t}{S_t}}$ is kept in the no-trading region $[\frac{1}{1+c}, \frac{1}{1+c/\bar{s}}]$.

The optimal growth rate

Definition

We call optimal growth rate the limit

$$\limsup_{T\to\infty} \frac{1}{T} E[\log X_T^{\theta^0,\theta}],$$

where (θ^0, θ) is the optimal strategy.

We have to compute $\limsup_{T\to\infty} \frac{1}{T} E[\int_0^T \frac{\tilde{\mu}^2(\frac{S_t}{m_t})}{2\tilde{\sigma}^2(\frac{S_t}{m_t})} dt]$. Define the stopping time $\tau = \inf\{t > 0 : \frac{S_t}{m_t} \leq 1\}$ and the measure $\nu(A) = \frac{1}{E[\tau]} E \int_0^\tau 1_A(\frac{S_t}{m_t}) dt$ for $A \in \mathcal{B}[1, \bar{s}]$.

The optimal growth rate

From Itô's formula we have

$$E\left[\varphi\left(\frac{S_t}{m_t}\right)\right] = \varphi(1) + E\int_0^t \mu\varphi'\left(\frac{S_t}{m_t}\right)\frac{S_t}{m_t} + \sigma^2\frac{\varphi''\left(\frac{S_t}{m_t}\right)\frac{S_t^2}{m_t^2}}{2}ds + E[\varphi'(0)L_t - \varphi'(\bar{s})U_t]$$

and then

$$\int \mu\varphi'(s)s + \sigma^2\frac{\varphi''(s)s^2}{2}\nu(ds) + \varphi'(0)l - \varphi'(\bar{s})u = 0.$$

Evaluating in $\varphi_1(s) = s^{1-\frac{2\mu}{\sigma^2}}$ and $\varphi_2(s) = \frac{1}{\mu-\sigma^2/2}\log s$ allows us to compute l and u. Assuming $\mu(ds) = p(s)ds$, by integration by parts we get conditions on $p(s)$, which lead to the following solution

$$\nu(ds) = \frac{2\pi - 1}{\bar{s}^{2\pi-1} - 1}s^{2\pi-2}1_{[1,\bar{s}]}(s)ds.$$

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$$+ E[\varphi'(0)L_t - \varphi'(\bar{s})U_t]$$

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$$\nu(ds) = \frac{2\pi - 1}{\bar{s}^{2\pi-1} - 1}s^{2\pi-2}1_{[1,\bar{s}]}(s)ds.$$

The optimal growth rate

The optimal growth rate is given by

$$\int_0^{\bar{s}} \frac{\tilde{\mu}^2(s)}{2\tilde{\sigma}^2(s)}\nu(ds) = \frac{\mu^2}{2\sigma^2} - \left(\frac{3\sigma^3}{2^{\frac{7}{2}}}\pi^2(1-\pi)^2\right)^{2/3}\lambda^{2/3} + O(\lambda^{4/3}).$$

Similar results can be obtained for the power utility case.

Part III

Liquidity

Motivation

Order book

Suppose that in the order book the quotes are distributed around the price S_t with density $\rho(x)$. If an agent wants to buy h units of stocks, it will have to pay up to a relative price s given by

$$h = \int_1^s \rho(x)dx.$$

There is no permanent effect of the trade on the price, so the loss is given by

$$Sl(h) = S\int_1^s x\rho(x)dx - hS = S\int_1^s (x-1)\rho(x)dx.$$

Motivation

Let us consider a time interval of length Δt, during which the relative quotes have density $\rho(x)dx\Delta t$ around S_t. If an agent wants to buy $h\Delta t$ units of stock, it will face a loss of $S_t l(h)\Delta t$.

In our framework we consider dynamics of the following type for the wealth process

$$dX_t = \theta_t dS_t - S_t l(\dot{\theta}_t)dt,$$

where $-S_t l(\dot{\theta}_t)dt$ represents the liquidity cost.

The problem

We assume that the wealth process has dynamics

$$dX_t = \theta_t dS_t - \lambda K_t(\dot{\theta}_t)^2 dt,$$

for a proper process K_t.

When $K_t = \frac{S_t}{\theta_t}$, the dynamics of X_t are

$$dX_t = X_t(\Pi_t(\mu dt + \sigma dW_t) - \lambda \Pi_t \left(\frac{\dot{\theta}_t}{\theta_t}\right)^2 dt).$$

When $K_t = \frac{S_t^2}{X_t}$, the dynamics of X_t are

$$dX_t = X_t(\Pi_t(\mu dt + \sigma dW_t) - \lambda \Pi_t^2 \left(\frac{\dot{\theta}_t}{\theta_t}\right)^2 dt).$$

H-J-B equation

We want to maximize $E[U(X_t^\theta)]$ for utilities with constant Relative Risk Aversion. Assume $K_t = \frac{S_t}{\theta_t}$.

The H-J-B equation is

$$\begin{aligned} v_t + \max_u \{ v_x \left(xy\mu - \lambda xyu^2 \right) \\ + v_y \left(y(1-y)(\mu - y\sigma^2) + yu + \lambda y^2u^2 \right) + \frac{1}{2} v_{xx} x^2y^2\sigma^2 \\ + \frac{1}{2} v_{yy} (y^2(1-y)^2\sigma^2) + v_{xy} xy^2(1-y)\sigma^2 \} = 0, \end{aligned}$$

with final condition $v(T,x,y) = U(x)$.

The control process is $\frac{\dot{\theta}_t}{\theta_t}$, the state processes are X_t and Π_t.

H-J-B equation

Consider the case $U(x) = \log x$.

Assume that the value function is of the form $v(t,x,y) = \beta(T-t) + z(y) + \log x$. This function does no longer satisfy the final condition, but will give us a candidate for a “long-run optimum”.

With such a value function, the corrisponding optimum control u has to satisfy the following Abel differential equation:

$$\begin{aligned}(-\beta + y\mu - \frac{1}{2}y^2\sigma^2) &+ (-4\beta + 2y\mu + 2\mu - 2y\sigma^2)y\lambda u \\ &+ (1 - 4\lambda\beta y + 4\lambda y\mu - 2\lambda y\sigma^2)\lambda y u^2 \\ &+ 2\lambda^2 y^2 u^3 + \lambda y^2(1-y)^2\sigma^2 u_y = 0.\end{aligned}$$

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Limit for small λ

Assume $\beta \approx \frac{\mu^2}{2\sigma^2} - c\lambda^\delta$. For a given λ, let u_λ be the solution of the Abel equation and define $w_\lambda = \sqrt{\lambda} u_\lambda$.
Then we get

$$w_0(y) = \frac{1}{\sqrt{2}\sigma} \left(\frac{\mu}{\sqrt{y}} - \sqrt{y}\sigma^2 \right).$$

Let y_λ the value such that $u_\lambda(y_\lambda) = 0$. If $y_\lambda = \pi + o(\lambda^{\frac{1}{4}})$, then by some continuity argument we get

$$\delta = \frac{1}{2}, \quad c = \frac{1}{\sqrt{2\pi}} \pi^2 (1-\pi)^2 \sigma^3.$$

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The optimal growth rate

When we consider $K_t = \frac{S_t^2}{X_t}$, we get

$$\delta = \frac{1}{2}, \quad c = \frac{1}{\sqrt{2}}\pi^2(1-\pi)^2\sigma^3.$$

Similar results are valid in the power utility case.

Long-run optimality

We still have to prove that our candidate function is indeed a long-run optimal strategy, i.e.

$$\limsup_{T\to\infty} \frac{E[U(X_T^\theta)]}{v(T,x,y)} = 1.$$

We would like to find a duality relation

$$E[U(X_T)] \leq v(T,x,y) \leq z(T,x,y)$$

such that the distance among $E[U(X_T)]$ and $z(T,x,y)$ gets narrow when $T \to \infty$.
Unfortunately, the martingale measure for X_t^θ depends on the strategy θ.

Comparison of impact of different frictions

Optimal growth rate

With transaction cost

$$\frac{\mu^2}{2\sigma^2} - \frac{3^{2/3}}{2^{7/3}} \left(\sigma^3\pi^2(1-\pi)^2\right)^{2/3} \lambda^{2/3}.$$

With liquidity cost

$$\frac{\mu^2}{2\sigma^2} - \frac{1}{2^{1/2}} \left(\sigma^3\pi^2(1-\pi)^2\right) \lambda^{1/2}.$$

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