Arbitrage-Free Pricing with Funding Costs and Collateralization

Andrea Pallavicini
a.pallavicini@imperial.ac.uk

Dept. of Mathematics, Imperial College London

Financial Engineering, Banca IMI

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Talk Outline

1. Securities, Derivatives and Trading Strategies
2. Arbitrage-Free Pricing
3. Funding Costs

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Talk Outline

1. Securities, Derivatives and Trading Strategies
   - Market Securities
   - Self-Financing Trading Strategies
   - Funding and Discounting
2. Arbitrage-Free Pricing
3. Funding Costs

Market Securities – I

- We start with the simple setting of a market with default-free securities.
  - $\longrightarrow$ We later add counterparty credit risk, funding costs and collateralization.
- We assume that the market quotes the prices of some securities we name $\{S_t^1, \ldots, S_t^n\}$.
- When holding a security we may face the possibility to receive or pay a quantity of cash.
  - $\longrightarrow$ The owner of a bond receives coupons on a regular basis.
  - $\longrightarrow$ Share holders receive dividends over time.
  - $\longrightarrow$ Many bilateral contracts consists in a strip of random cash flows.

Market Securities – II

- We name $\{\gamma_{T_1}, \ldots, \gamma_{T_N}\}$ the coupons, dividends or cash flows received or paid while holding a security
- We define the cumulative dividend process as

$$D_t := \sum_{i=1}^{N} \gamma_{T_i} 1_{\{T_i \leq t\}}$$

- The profits and losses achieved holding a security are described by the gain process, which is defined as

$$G_t := S_t + D_t$$

Trading Portfolios and Total Wealth – I

- A trading strategy in the market securities consists in holding a portfolio of securities.
- We name $\{q_t^1, \ldots, q_t^n\}$ the quantities of each security held in the portfolio.
  - $\longrightarrow$ At each time the trader may change the composition of the portfolio.
  - $\longrightarrow$ The quantities q_t^i may be either positive or negative.
- The total wealth realized by the strategy can be computed by taking into account the profit and losses along time.
- If we can trade only on times $\{t_0 = 0, t_1, \ldots, t_m = t\}$, we can write the total wealth as

$$W_t := \sum_{i=1}^{n} q_{t_0}^i S_{t_0}^i + \sum_{k=1}^{m} \sum_{i=1}^{n} q_{t_{k-1}}^i (G_{t_k}^i - G_{t_{k-1}}^i)$$

Trading Portfolios and Total Wealth – II

- We can substitute the definition of gain process in the total wealth formula to highlight how the dividends contribute to it.

$$W_t = \sum_{i=1}^{n} q^i_{t_0} S^i_{t_0} + \sum_{k=1}^{m} \sum_{i=1}^{n} q^i_{t_{k-1}} \left(S^i_{t_k} - S^i_{t_{k-1}} + \sum_{i=1}^{N} \gamma_{T_i} 1_{\{t_{k-1} < T_i \leq t_k\}} \right)$$

- A simple example of trading strategy is entering a position and never changing it, namely q_t does not depend on time.
  - $\longrightarrow$ In this case we should obtain that the total wealth is simply the sum of the gain processes of each security times their quantities.

Trading Portfolios and Total Wealth – III

- The wealth of a constant-quantity trading strategy

$$
\begin{aligned}
W_t &\doteq \sum_{i=1}^{n} q^i S^i_{t_0} + \sum_{i=1}^{n} q^i \sum_{k=1}^{m} \left(S^i_{t_k} - S^i_{t_{k-1}} + \sum_{i=1}^{N} \gamma_{T_i} 1_{\{t_{k-1} < T_i \leq t_k\}} \right) \\
&= \sum_{i=1}^{n} q^i S^i_{t_0} + \sum_{i=1}^{n} q^i \left(S^i_{t_m} - S^i_{t_0} + \sum_{i=1}^{N} \gamma_{T_i} 1_{\{T_i \leq t_m\}} \right) \\
&= \sum_{i=1}^{n} q^i \left(S^i_{t_m} + D_{t_m} \right) \\
&= \sum_{i=1}^{n} q^i G^i_t
\end{aligned}
$$

Self-Financing Trading Strategies – I

- An interesting class of trading strategies is given by the self-financing strategies.
- The wealth process of a self financing strategy is always equal to the liquidation value of the portfolio.

$$W_t \doteq \sum_{i=1}^{n} q_t^i S_t^i$$

- Which is the consequence of such constraint on the quantities q_t ?

Self-Financing Trading Strategies – II

- We focus on the increment in the wealth process over time.

$$W_{t_k} - W_{t_{k-1}} = \sum_{i=1}^{n} q^i_{t_{k-1}} (S^i_{t_k} - S^i_{t_{k-1}} + D^i_{t_k} - D^i_{t_{k-1}})$$

- If we require that the strategy is self-financing, we get

$$W_{t_k} - W_{t_{k-1}} = \sum_{i=1}^{n} (q^i_{t_k} S^i_{t_k} - q^i_{t_{k-1}} S^i_{t_{k-1}})$$

- If we equate the two expressions, we obtain

$$\sum_{i=1}^{n} q^i_{t_k} S^i_{t_k} = \sum_{i=1}^{n} q^i_{t_{k-1}} \left(S^i_{t_k} + D^i_{t_k} - D^i_{t_{k-1}} \right)$$

Self-Financing Trading Strategies – III

- Thus, the quantities are selected so that
  - $\longrightarrow$ dividends are re-invested in the strategy;
  - $\longrightarrow$ further cash is not required and no cash outflow is generated.
- In this sense the strategy is self-financing.
- Some examples are:
  - $\longrightarrow$ A strategy in shares of a company. Every time a dividend is paid the trader must buy more shares. This strategy is self-financing.
  - $\longrightarrow$ A strategy in a zero-coupon bond. At maturity the zero-coupon bond pays the notional, but we cannot re-invest in it since the contract is terminated. We need a second security to build a self-financing strategy.

Trading Strategies in Continuous Time

- In the following we use a continuous-time notation, and we express the cumulative dividend process as

$$D_t := D_0 + \int_0^t d\pi_u \;, \quad \pi_t := \sum_{i=1}^{N} \gamma_{T_i} 1_{\{T_i \leq t\}}$$

while the wealth process for the trading strategy q_t is given by

$$W_t := q_0 \cdot S_0 + \int_0^t q_u \cdot dG_u$$

where the internal products is in security space. If the strategy is self-financing we write

$$W_t \doteq q_t \cdot S_t$$

Treasury Bank Account – I

- Implement a trading strategies we need to access some cash-paying (and cash-receiving) securities to fund (and to invest) dividends.
  - $\longrightarrow$ For instance, if we have to pay at a future time T a unit of cash, we can buy a zero-coupon bond paying such cash at T.
- Since trading strategies have their own trading horizons, we wish to access cash-paying (and cash-receiving) securities without a maturity time.
- In practice we need a bank account.
  - $\longrightarrow$ We can enter into a bank account by paying one unit of cash at inception, and receiving it back at any later time along with a compensation.
  - $\longrightarrow$ On the other hand, we can also get one unit of cash at inception to pay it back at a later time along with a fee.
- Do bank accounts exist in the market ?

Treasury Bank Account – II

- On the market we have saving accounts, but their are intended for retail operations.
- Traders may access a special bank account, named the Treasury Bank Account (TBA), which is managed by the bank treasury department.
  - ⟶ The TBA is not a real security traded on the market, but it behaves as a security from the point of view of traders.
  - ⟶ The TBA is implemented by the treasury by issuing bonds, using collateral portfolios, accessing saving accounts, etc. . .
- The compensation rate, received when borrowing cash, and the fees, required when lending cash, are decided by the treasury.

Treasury Bank Account – III

- If we assume that the lending and borrowing rates are the same, name them r_t, we can calculate the price process B_t of the TBA as the solution of

$$dB_t = r_t B_t \, dt \,, \quad B_0 = 1$$

namely

$$B_t = \exp\left\{ \int_0^t du \, r_u \right\}$$

- In the following we assume that the TBA is one of the security used to implement trading strategies.

  $\longrightarrow$ We discuss again this assumption when funding costs are introduced.

Price Deflators

- When we say that the price process of a security is given by S_t we are thinking of liquidating the security to obtain an amount of cash equal to S_t.

  $\longrightarrow$ Cash behaves as a unit of measure for prices.

- Yet, we cannot access cash without paying fees or receiving compensations, since we lend and borrow cash by means of the TBA.
- Thus, to take into account the cost of money, we need to express the wealth processes in term of the TBA, namely

$$\bar{W}_t := \frac{W_t}{B_t}$$

  where $\bar{W}$ is the deflated wealth.

- How can we define deflated price and cumulative dividend processes ?

Invariance of Self-Financing Trading Strategies – I

- We require that the property of a trading strategy of being self-financing is invariant under deflation.
  - ⟶ We define the deflated price and cumulative dividend processes to ensure this property.
- If q_t is a self-financing strategy $(W_t \doteq q_t \cdot S_t)$ we can write

$$\bar{W}_t = \frac{W_t}{B_t} = q_t \cdot \bar{S}_t$$

  where we define the deflated price process as

$$\bar{S}_t := \frac{S_t}{B_t}$$

- The definition of the deflated cumulative dividend process is less obvious, since we must consider that dividends are paid over time, and the TBA value depends on time too.

Invariance of Self-Financing Trading Strategies – II

- Starting from the definition of deflated wealth, we can write

$$
\begin{aligned}
\bar{W}_t &= \bar{W}_0 + \int_0^t \left(\frac{dW_u}{B_u} - W_u r_u B_u \, du \right) \\
&= q_0 \cdot S_0 + \int_0^t q_u \cdot \left(\frac{dG_u}{B_u} - S_u r_u B_u \, du \right) \\
&= q_0 \cdot S_0 + \int_0^t q_u \cdot \left(\frac{dS_u}{B_u} - S_u r_u B_u \, du + \frac{dD_u}{B_u} \right) \\
&= q_0 \cdot S_0 + \int_0^t q_u \cdot d\bar{G}_u
\end{aligned}
$$

where we define the deflated cumulative dividend and gain processes

$$
\bar{D}_t := D_0 + \int_0^t \frac{dD_u}{B_u} \,, \quad \bar{G}_t := \bar{S}_t + \bar{D}_t
$$

Invariance of Self-Financing Trading Strategies – III

- If the bank account is risky, as in a foreign-currency account, the definition of the deflated processes must take into account the covariation of the dividend process with the deflator.
- For a generic positive process Y_t (deflator) we can follow Duffie (2001) to write:

$$W_t^Y = q_0 \cdot S_0^Y + \int_0^t q_u \cdot dG_u^Y \ , \quad G_t^Y := S_t^Y + D_t^Y$$

where we define the deflated price and cumulative dividend processes

$$S_t^Y := Y_t S_t \ , \quad D_t^Y := Y_0 D_0 + \int_0^t (Y_u \, dD_u + d\langle Y, D \rangle_u)$$

Talk Outline

Arbitrages – I

- In efficient markets securities are always traded at their fair value.
  - $\longrightarrow$ Investors can possibly obtain higher returns only by purchasing riskier investments.
- The possibility "to make money from nothing without risks" should be excluded from the set of possible trading strategies.
  - $\longrightarrow$ We name arbitrages such strategies.
- A more formal definition of arbitrage is needed to going on.
- We refer again to Duffie (2001) for the huge literature on arbitrages and their relationship with martingale pricing.

Arbitrages – II

- We introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with
  - $\longrightarrow$ the standard filtration $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$ generated by the security price processes, and
  - $\longrightarrow$ the physical probability measure $\mathbb{P}$ representing the actual distribution of supply-and-demand shocks on security prices.
- We can define arbitrages as a self-financing trading strategy q_t whose wealth at inception time t is non-positive, namely

$$W_t \leq 0$$

while at maturity T it is never negative, and it is strictly positive in some state, so that we can write

$$W_T \geq 0 \,, \quad \mathbb{P}\{\, W_T > 0 \,\} > 0$$

- To avoid arbitrages we can impose some conditions on the wealth process W.

Equivalent Martingale Pricing – I

- Given the TBA as price deflator, we can ensure the absence of arbitrages, if we can find a measure $\mathbb{Q}$, equivalent to the physical measure $\mathbb{P}$, such that the deflated gain process $\bar{G}_t$ is a martingale under such measure.

  $\longrightarrow$ The measure $\mathbb{Q}$ is known as risk-neutral measure.
- Arbitrages are forbidden even if we use a generic deflator Y_t.

  $\longrightarrow$ In this case the measure $\mathbb{Q}^Y$ depends on the choice of the deflator, and it is known as equivalent martingale measure.
- The reverse is not true in general.

Equivalent Martingale Pricing – II

- Under suitable technical conditions on the trading strategy q_t, the martingale condition allows us to write

$$\mathbb{E}\big[\,\bar{W}_T \mid \mathcal{F}_t\,\big] = \bar{W}_t + \int_t^T \mathbb{E}\big[\,q_u \cdot d\bar{G}_u \mid \mathcal{F}_t\,\big] = \bar{W}_t$$

  where the expectations are taken under the risk-neutral measure.

- If q_t is an arbitrage, we have $W_t \leq 0$ and

$$W_T \geq 0 \implies \bar{W}_T \geq 0 \implies \bar{W}_t = \mathbb{E}\big[\,\bar{W}_T \mid \mathcal{F}_t\,\big] \geq 0$$

  on the other hand, the equivalence between the measures implies

$$\mathbb{P}\{\,W_T > 0\,\} > 0 \implies \mathbb{Q}\{\,W_T > 0\,\} > 0 \implies \mathbb{Q}\big\{\,\bar{W}_T > 0\,\big\} > 0$$

  leading to $\bar{W}_t > 0$ which contradicts the hypothesis.

Equivalent Martingale Pricing – III

- If we assume the existence of a risk-neutral measure, we can price market securities with maturity date T by exploiting the martingale condition of deflated gain processes.

$$\bar{G}_t = \mathbb{E}\left[\, \bar{G}_T \;|\, \mathcal{F}_t \,\right]$$

- Then, we can expand the gain process to obtain the arbitrage-free pricing formula under $\mathbb{Q}$-expectation

$$S_t = B_t \,\mathbb{E}\left[\frac{S_T}{B_T} + \int_t^T \frac{dD_u}{B_u} \;|\, \mathcal{F}_t \right]$$

or for a generic deflator Y_t under $\mathbb{Q}^Y$-expectation

$$S_t = \frac{1}{Y_t}\mathbb{E}^Y\left[Y_T S_T + \int_t^T \left(Y_u \, dD_u + d\langle Y, D\rangle_u \right) \,|\, \mathcal{F}_t \right]$$

Replication of Derivative Contracts – I

- We can extend pricing formulae to derivative securities not traded on the market.
- We consider a derivative with price process V_t and cumulative dividend process Q_t.
- In order to replicate the derivative in terms of market securities, we can implement a strategy q_t to invest (or to fund) the dividend s received (or paid) by the derivative, namely

$$Q_t \doteq W_t - q_t \cdot S_t$$

- Furthermore, we require that at maturity the price of the constituents of the strategy is equal to the price of the derivative.

$$V_T \doteq q_T \cdot S_T$$

Replication of Derivative Contracts – II

- The derivative price can be calculated at any time from the market security prices.
- We consider a trading strategy q' which invests in the market securities as the strategy q_t and shorts one unit of the derivative, namely
$$q'_t := (q_t, -1)$$
- The wealth generated by such strategy is given by
$$W'_t = q_0 \cdot S_0 - V_0 + \int_0^t (q_u \cdot dG_u - dV_u - dQ_u) = q_t \cdot S_t - V_t$$
so that we can conclude that the strategy q' is self-financing with null final wealth, $W'_T = 0$.

Replication of Derivative Contracts – III

- If we require absence of arbitrages, we obtain that at any time $t < T$ we must have

$$W'_T \geq 0 \implies W'_t \geq 0 \implies q_t \cdot S_t \geq V_t$$

On the other hand, we can consider the strategy $(-q_t, 1)$ leading to

$$q_t \cdot S_t \leq V_t$$

Thus, we have at any time t up to matuirty T that

$$V_t = q_t \cdot S_t$$

- We can write that the derivative gain process is equal to the wealth generated by the replicating strategy q_t.

$$W_t = V_t + Q_t$$

Replication of Derivative Contracts – IV

- If we assume the existence of a risk-neutral measure for the market securities, we have that the deflated gain process of the derivative is a martingale too, leading to the pricing equation

$$V_t = B_t \, \mathbb{E}\left[\frac{V_T}{B_T} + \int_t^T \frac{dQ_u}{B_u} \,|\, \mathcal{F}_t \right]$$

or for a generic deflator Y_t under $\mathbb{Q}^Y$-expectation

$$V_t = \frac{1}{Y_t} \mathbb{E}^Y \left[Y_T V_T + \int_t^T \left(Y_u \, dQ_u + d\langle Y, Q \rangle_u \right) \,|\, \mathcal{F}_t \right]$$

which can be solved once a terminal condition for V_T is selected.

Market and Enlarged Filtrations – I

- The next element we add to the pricing framework is the possibility of default of one of the counterparties of the contract.
- How can we deal with the default event under the risk-neutral measure?
  - $\longrightarrow$ We need to describe the filtration to adopt to calculate the risk-neutral expectations.
- Market risks for contracts with defaultable counterparties arise from the uncertainty both in default probabilities and in the default times.
  - $\longrightarrow$ We could add risks specific of the underlying asset and recoveries as well.
- As a first step we introduce the market filtration $\mathcal{F}_t$ representing all the observable market quantities but the default events.

Market and Enlarged Filtrations – II

- Then, we define the default events of the counterparty τ_C and of the investor τ_I along with the first default time

$$\tau := \tau_C \wedge \tau_I$$

- We define the enlarged filtration $\mathcal{G}$ containing also the default monitoring.

  $\longrightarrow$ See Bielecki and Rutkowski (2001) for details.

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}^C_t \vee \mathcal{H}^I_t \supseteq \mathcal{F}_t$$

$$\mathcal{H}^k_t := \sigma(\{\tau_k \leq u\} : u \leq t)\,, \quad k \in \{C, I\}$$

Market and Enlarged Filtrations – III

- From the definition of $\mathcal{G}$, we can write

$$\forall g_t \in \mathcal{G}_t \; \exists f_t \in \mathcal{F}_t : g_t \cap \{\tau_C > t\} \cap \{\tau_I > t\} = f_t \cap \{\tau_C > t\} \cap \{\tau_I > t\}$$

  or simply

$$\forall g_t \in \mathcal{G}_t \; \exists f_t \in \mathcal{F}_t : g_t \cap \{\tau > t\} = f_t \cap \{\tau > t\}$$

- Thus, for any $\mathcal{G}$-adapted process x_t we can introduce the pre-default $\mathcal{F}$-adapted process $\tilde{x}_t$ such that

$$1_{\{\tau > t\}} x_t = 1_{\{\tau > t\}} \tilde{x}_t$$

- We use this property for numerical implementations to express expectations under the enlarged $\mathcal{G}$ filtration as expectations under the market $\mathcal{F}$ filtration.

Trading Strategies with Defaultable Counterparties – I

- The counterparty credit risk is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction cash flows.
  - $\longrightarrow$ When one of the counterparty defaults the trade is terminated.
  - $\longrightarrow$ An economic loss would occur if the transaction with the counterparty has a positive economic value at the time of default.
- We can accommodate counterparty risk by terminating the dividend process at the first default event, and setting the terminal condition for the security price accordingly.

$$S_{T\wedge\tau} := 1_{\{\tau\leq T\}}\theta_\tau \ , \quad D_t := D_0 + \int_0^t 1_{\{\tau>u\}} d\pi_u$$

where θ_τ is the cash flow paid if the default occurs, and without loss of generality we set $1_{\{\tau>T\}}S_T \doteq 0$.

Trading Strategies with Defaultable Counterparties – II

- To avoid arbitrages we require that the deflated gain processes are martingale under the $\mathcal{G}$ filtration.
- The pricing equation becomes

$$S_t = B_t \, \mathbb{E}\left[1_{\{\tau \leq T\}} \frac{\theta_\tau}{B_\tau} + \int_t^T 1_{\{\tau > u\}} \frac{dD_u}{B_u} \mid \mathcal{G}_t \right]$$

- A similar expression holds for generic deflators Y_t.
- Since credit default risk introduces an element of non-predictability, we cannot implement a replication strategy to price derivative securities, but in simple cases.

  $\longrightarrow$ However, we can price them as any other market security.

Close-Out Netting Rules – I

- In case of default of one party, the surviving party should evaluate the transactions just terminated, due to the default event occurrence, to claim for a reimbursement after the application of netting rules to consolidate the transactions.

  $\longrightarrow$ The amount of the cash flow θ_τ results from such analysis.

- The cash flow θ_τ is described by the ISDA documentation as given by

$$\begin{aligned}\theta_\tau &:= 1_{\{\tau_C<\tau_I\}}\left(R_C\varepsilon_\tau^+ + \varepsilon_\tau^-\right) + 1_{\{\tau_I<\tau_C\}}\left(\varepsilon_\tau^+ + R_I\varepsilon_\tau^-\right) \\ &= \varepsilon_\tau - 1_{\{\tau_C<\tau_I\}}(1-R_C)\varepsilon_\tau^+ + 1_{\{\tau_I<\tau_C\}}(1-R_I)\varepsilon_\tau^-\end{aligned}$$

  where R_C and R_I are the recovery rates, and ε_τ is the close-out amount representing the exposure measured by the surviving party on the default event.

Close-Out Netting Rules – II

- It is difficult to define the close-out amount, and also ISDA is not very assertive on the topic.

  $\longrightarrow$ See Brigo, Morini and Pallavicini (2013) for a review.

- You may have a risk free close-out, where the residual deal is priced at mid market without any residual counterparty risk.

$$\varepsilon_\tau \doteq B_\tau \int_{\tau\wedge T}^{T} \mathbb{E}\left[\frac{d\pi_u}{B_u} \,|\, \mathcal{G}_\tau \right]$$

- You may have a replacement close-out, where the remaining deal is priced by taking into account the credit quality of the surviving party and of the party that replaces the defaulted one.
- A possible guess for the pre-default close-out is given by

$$\varepsilon_\tau \doteq \widetilde{S}_{\tau^-}$$

Close-Out Netting Rules – III

- The replacing pre-default close-out is the first example of non-linearities in the pricing equation.
- Indeed, if we write the pre-default price we get

$$1_{\{\tau>t\}}\widetilde{S}_t = 1_{\{\tau>t\}} B_t \, \mathbb{E}\left[1_{\{\tau\leq T\}} \frac{\theta_\tau(\widetilde{S}_\tau)}{B_\tau} + \int_t^T 1_{\{\tau>u\}} \frac{dD_u}{B_u} \,|\, \mathcal{G}_t \right]$$

- The above expression is an implicit equation for the the pre-default price of the security, which could be without solutions.
- In the following, when we introduce collateralization and funding costs, we discuss again such problem.

Collateralization and Counterparty Credit Risk

- The growing attention on counterparty credit risk is transforming OTC derivatives money markets:
  - $\longrightarrow$ an increasing number of derivative contracts is cleared by CCPs, while
  - $\longrightarrow$ most of the remaining contracts are traded under collateralization.
- Both cleared and bilateral deals require collateral posting, along with its remuneration.
- Collateralized bilateral trades are regulated by ISDA documentation, known as Credit Support Annex (CSA).
- Centralized clearing is regulated by the contractual rules described by each CCP documentation.
- See Brigo et al. (2012) and Brigo and Pallavicini (2014) for a description of bilateral-traded and centrally-cleared contracts.

Trading Strategies with Margining Procedures – I

- We can include the margining procedure within arbitrage-free pricing by extending the definition of the gain and the cumulative dividend process.
- In general, a margining practice consists in a pre-fixed set of dates during the life of a deal when both parties post or withdraw collaterals, according to their current exposure, to or from an account held by the Collateral Taker.
- We consider that a positive collateral account C_t is held by the investor, otherwise by the counterparty. Moreover, as we set a null terminal condition for the security price, we set $C_T \doteq 0$.
- The Collateral Taker remunerates the account at rate c_t fixed by the collateralization agreement.

  $\longrightarrow$ The collateral rate may depend on the sign of the collateral account.

Trading Strategies with Margining Procedures – II

- Thus, the cumulative dividend process can be extended in the following way

$$D_t := D_0 + \int_0^t 1_{\{\tau > u\}} \left(d\pi_u + dC_u - c_u C_u \, du \right)$$

- Notice that including the collateral account in the cumulative dividend process means that we can re-hypothecate its content.
- Moreover, at trade termination we have to withdraw collateral assets kept in our accounts, so that the gain process can be re-defined as

$$G_t := S_t + D_t - C_t$$

Trading Strategies with Margining Procedures – III

- To avoid arbitrages we require that the deflated gain processes are martingale under the $\mathcal{G}$ filtration.
- Thus, we get

$$\bar{G}_t = \mathbb{E}\left[\,\bar{G}_{T\wedge\tau}\,|\,\mathcal{G}_t\right] \implies \bar{S}_t = \bar{C}_t + \mathbb{E}\left[\bar{S}_{T\wedge\tau} - \bar{C}_{T\wedge\tau} + \int_t^T 1_{\{\tau>u\}}\,d\bar{D}_u\,|\,\mathcal{G}_t\right]$$

- The integral over deflated dividends can be written as

$$\begin{aligned}\int_t^T 1_{\{\tau>u\}}\,d\bar{D}_u &= \int_t^T 1_{\{\tau>u\}}\left(\frac{d\pi_u}{B_u} + \frac{dC_u}{B_u} - \frac{c_u C_u\,du}{B_u}\right) \\ &= \frac{C_{T\wedge\tau}}{B_{T\wedge\tau}} - \frac{C_{t\wedge\tau}}{B_{t\wedge\tau}} + \int_t^T 1_{\{\tau>u\}}\left(\frac{d\pi_u}{B_u} + \frac{(r_u - c_u)C_u\,du}{B_u}\right)\end{aligned}$$

Trading Strategies with Margining Procedures – IV

- If we substitute the expression for the dividend integral, we get the pricing equation

$$1_{\{\tau>t\}}\widetilde{S}_t = 1_{\{\tau>t\}} B_t \, \mathbb{E}\left[1_{\{\tau\leq T\}} \frac{\theta_\tau}{B_\tau} + \int_t^T 1_{\{\tau>u\}} \left(\frac{d\pi_u}{B_u} + \frac{(r_u - c_u) C_u \, du}{B_u} \right) \,|\, \mathcal{G}_t \right]$$

- According to ISDA the definition of the on-default cash flow in presence of collateralization and re-hypothecation is given by

$$\theta_\tau := \varepsilon_\tau - 1_{\{\tau_C<\tau_I\}}(1-R_C)(\varepsilon_\tau - C_{\tau^-})^+ - 1_{\{\tau_I<\tau_C\}}(1-R_I)(\varepsilon_\tau - C_{\tau^-})^-$$

- Another source of non-linearities occurs if the collateral account is proportional to the pre-default price of the derivative.

$$C_t \doteq \alpha_t \widetilde{S}_t$$

where α_t is a $\mathcal{F}$-adapted process.

Talk Outline

How to Construct a Bank Account

- When we derive the pricing equations, we assume the availability of a treasury bank account.
- Now, we analyse how it is implemented by the treasury, and if counterparty risk may change this construction.
- Bank accounts are used by traders both for cash lending and borrowing.
  - $\longrightarrow$ Trading strategies to borrow and to lend cash are differently implemented, leading to different bank accounts.
  - $\longrightarrow$ See Bergman (1995), Crépey (2011), Pallavicini, Perini and Brigo (2011).
- We consider the following stylized procedure up to time t.
  - $\longrightarrow$ Lending: a trading desk has a surplus of cash to be invested at time 0, at time t the desk gets the cash back with a premium.
  - $\longrightarrow$ Borrowing: a trading desk needs cash at time 0, at time t the desk gives the cash back with a fee.

Lending Bank Account – I

- We start by discussing the lending case.
- In particular, we assume that a bank "I" invests cash in zero-coupon bonds of a counterparty "C".
- Along with the position in bonds the bank shall buy protection for losses due to the default of the counterparty.
  - $\longrightarrow$ The bank can buy a Credit Default Swap (CDS) for each bond in the strategy.
  - $\longrightarrow$ A CDS contract protects the bond owner from losses occurring on default time by paying a fee s^{l}_{t}.
- If the counterparty defaults the CDS covers all losses, and the bank may open a new position with another counterparty.
- The strategy can be implemented up to time t or up to the default of the bank. In particular, we assume to roll the positions on a time grid

$$\{t_0 = 0, t_1, \ldots, t_m = t\}$$

Lending Bank Account – II

- At time t_0 the bank buys a zero-coupon bond of the counterparty with maturity t_1 and notional

$$q^{\mathrm{l}}_{t_0} := \frac{1}{P^{\mathrm{l}}_{t_0}(t_1)}$$

where $P^{\mathrm{l}}_{t_0}(t_1)$ is the bond market price, so that we have a cash flow of

$$1_{\{\tau > t_0\}} \gamma^{\mathrm{buy}}_{t_0} := -1_{\{\tau > t_0\}} q^{\mathrm{l}}_{t_0} P^{\mathrm{l}}_{t_0}(t_1)$$

- At the same time the bank enters at par into a CDS contract with maturity t_1 on the same bond.

Lending Bank Account – III

- At time t_1 the notional of the bond is returned to the bank and the CDS fee is paid, if neither the bank nor the counterparty has defaulted between t_0 and t_1.

$$1_{\{\tau>t_1\}}\gamma_{t_1}^{\mathrm{receive}} := 1_{\{\tau>t_1\}}q_{t_0}^{\mathrm{l}} \,, \quad 1_{\{\tau>t_1\}}\gamma_{t_1}^{\mathrm{fee}} := -1_{\{\tau>t_1\}}q_{t_0}^{\mathrm{l}}s_t^{\mathrm{l}}(t_1-t_0)$$

- If a default happens, and the defaulting party is the counterparty, the CDS covers all losses, and on the next time-step the position is opened with another counterparty.
- If the bank survives, all contracts are opened again with notional

$$q_{t_1}^{\mathrm{l}} := \frac{q_{t_0}^{\mathrm{l}}(1-s_t^{\mathrm{l}}(t_1-t_0))}{P_{t_1}^{\mathrm{l}}(t_2)}$$

so to build a self-financing strategy, namely

$$\gamma_{t_1}^{\mathrm{receive}}+\gamma_{t_1}^{\mathrm{fee}}+\gamma_{t_1}^{\mathrm{buy}}=0$$

Lending Bank Account – IV

- Thus, we can sum all the contributions up to time t, or up to the default of the bank, to define the wealth generated by the investing strategy.

$$
\begin{aligned}
W^{\mathrm{l}}_{t\wedge\tau_I} &:= 1 + \sum_{k=0}^{m-1} 1_{\{\tau_I > t_k\}} \gamma^{\mathrm{buy}}_{t_k} + \sum_{k=1}^{m} 1_{\{\tau_I > t_k\}} \left(\gamma^{\mathrm{receive}}_{t_k} + \gamma^{\mathrm{fee}}_{t_k}\right) \\
&= \prod_{k=1}^{m} 1_{\{\tau_I > t_k\}} \frac{1 - s^{\mathrm{l}}_{t_k}(t_k - t_{k-1})}{P^{\mathrm{l}}_{t_{k-1}}(t_k)}
\end{aligned}
$$

- We can write the wealth of the strategy in continuous time as

$$
W^{\mathrm{l}}_{t\wedge\tau_I} = \exp\left\{ \int_0^{t\wedge\tau_I} du \, \left(y^{\mathrm{l}}_u - s^{\mathrm{l}}_u\right) \right\} , \quad y^{\mathrm{l}}_t := -\partial_T \log P^{\mathrm{l}}_t(T)|_{T=t}
$$

where y^{l}_t is the market yield of the bond issued by the counterparty.

Lending Bank Account – V

- Up to the default of the bank (included) the wealth process is a locally risk-free bank account, independently of the counterparty issuing the bonds.
- All these accounts are derived securities, so that, to avoid arbitrages, all the bond/CDS bases must be equal to the same rate r_t.

$$r_t := y_t^{\mathrm{l}} - s_t^{\mathrm{l}}$$

- In the practice many factors, like bond and CDS market liquidity, CDS collateralization and gap risk, default event specification, etc. . . , prevent to extract r_t from bond and CDS quotes.
- For later convenience, we cast the bond/CDS basis as a spread ℓ_t^{l} over the overnight rate e_t, and we write

$$\ell_t^{\mathrm{l}} := y_t^{\mathrm{l}} - s_t^{\mathrm{l}} - e_t$$

Borrowing Bank Account – I

- We continue the discussion with the borrowing case.
- In particular, we assume that a bank "I" obtains cash by issuing zero-coupon bonds.
- Notice that the bank cannot buy protection on herself to hedge its own default event.
- At time t_0 the bank issues a zero-coupon bond with maturity t_1 and notional

$$q^{\mathrm{b}}_{t_0} := \frac{1}{P^{\mathrm{b}}_{t_0}(t_1)}$$

where $P^{\mathrm{b}}_{t_0}(t_1)$ is the bond market price, so that we have a cash flow of

$$1_{\{\tau_I > t_0\}} \gamma^{\mathrm{issue}}_{t_0} := 1_{\{\tau_I > t_0\}} q^{\mathrm{b}}_{t_0} P^{\mathrm{b}}_{t_0}(t_1)$$

Borrowing Bank Account – II

- If the bank defaults, the strategy is terminated and the bond owner recovers only a fraction R_I of the notional.

$$1_{\{t_0<\tau_I\le t_1\}}\gamma^{\text{recovery}}_{\tau_I} := -1_{\{t_0<\tau_I\le t_1\}} R_I q^{\text{b}}_{t_0}$$

- If the bank survives, at time t_1 the notional of the bond is returned to the counterparty.

$$1_{\{\tau_I>t_1\}}\gamma^{\text{pay}}_{t_1} := -1_{\{\tau_I>t_1\}} q^{\text{b}}_{t_0}$$

and all contracts are opened again with notional

$$q^{\text{b}}_{t_1} := \frac{q^{\text{b}}_{t_0}}{P^{\text{b}}_{t_1}(t_2)}$$

so to build a self-financing strategy (but on bank default event), namely

$$\gamma^{\text{pay}}_{t_1} + \gamma^{\text{issue}}_{t_1} = 0$$

Borrowing Bank Account – III

- Thus, we can sum all the contributions up to time t, or up to the default of the bank, to define the wealth generated by the funding strategy.

$$\begin{aligned} W^{\mathrm{b}}_{t\wedge\tau_I} &:= -1 + \sum_{k=0}^{m-1} 1_{\{\tau_I > t_k\}} \left(\gamma^{\mathrm{issue}}_{t_k} + 1_{\{\tau_I \leq t_{k+1}\}}\gamma^{\mathrm{recovery}}_{\tau_I}\right) + \sum_{k=1}^{m} 1_{\{\tau_I > t_k\}}\gamma^{\mathrm{pay}}_{t_k} \\ &= -\prod_{k=1}^{m} 1_{\{\tau_I > t_k\}} \frac{1}{P^{\mathrm{b}}_{t_{k-1}}(t_k)} - R_I \sum_{k=0}^{m-1} 1_{\{t_k < \tau_I \leq t_{k+1}\}} \prod_{j=1}^{k} \frac{1}{P^{\mathrm{b}}_{t_{j-1}}(t_j)} \end{aligned}$$

- We can write the wealth of the strategy in continuous time as

$$W^{\mathrm{b}}_{t\wedge\tau_I} = \widetilde{W}^{\mathrm{b}}_{t\wedge\tau_I} - 1_{\{t=\tau_I\}}(1-R_I)\widetilde{W}^{\mathrm{b}}_{\tau_I} \,, \quad \widetilde{W}^{\mathrm{b}}_t := -\exp\left\{\int_0^t du\, y^{\mathrm{b}}_u\right\}$$

where y^{b}_t is the market yield of the bond issued by the bank.

Borrowing Bank Account – IV

- Only up to the default of the bank (excluded) the wealth process is a locally risk-free bank account.
- On bank default the position is terminated with an additional cash flow, given by

$$(1 - R_I) \exp\left\{ \int_0^{\tau_I} du \, y_u^{\mathrm{b}} \right\}$$

- We notice that, in case of default of the bank, we have a funding benefit since only a part of the reimbursement will be fulfilled.

  $\longrightarrow$ See Crépey (2011) and Pallavicini, Perini and Brigo (2011).
- For later convenience, we express the yield of bank bonds as a spread ℓ_t^{b} over the overnight rate e_t, and we write

$$\ell_t^{\mathrm{b}} := y_t^{\mathrm{b}} - s_t^{\mathrm{b}} - e_t$$

  where s_t^{b} is the CDS spread of the bank.

Trading Strategies with Funding Costs – I

- Lending and borrowing strategies are used by the treasury to assist trading activities.
- We can focus on a particular trading strategy in market or derived securities which is funded by the treasury on a netting base (funding netting set).
- The assignment of a security to a particular netting set is decided by the treasury.
  - ⟶ A possible choice is a netting set including all the trades of the bank.
  - ⟶ We assume that contracts of the same counterparty are not split among different netting sets.
  - ⟶ See Pallavicini, Perini and Brigo (2011), Albanese and Andersen (2015).
- Now, we try to establish a pricing formula for the whole netting set. This exercise requires to re-define the TBA to take into account the possibility of default of the bank.

Trading Strategies with Funding Costs – II

- We can accommodate the funding benefit by setting the terminal condition on bank default as given by

$$S_{T\wedge\tau} := 1_{\{\tau\leq T\}}\theta_\tau + 1_{\{\tau=\tau_I<T\}}(q^f_{\tau_I})^+(1-R_I)B^f_{\tau_I}$$

where q^f_t is the quantity of cash allocated by the treasury to fund the security S_t within the netting set, and the TBA is defined as

$$B^f_t := 1_{\{q^f_t>0\}}B^{\mathrm{b}}_t + 1_{\{q^f_t\leq 0\}}B^{\mathrm{l}}_t$$

where the lending and borrowing bank accounts are defined as

$$B^{\mathrm{l}}_t := W^{\mathrm{l}}_t = \exp\left\{\int_0^t du\,\left(y^{\mathrm{l}}_u - s^{\mathrm{l}}_u\right)\right\}\ , \quad B^{\mathrm{b}}_t := -\widetilde{W}^{\mathrm{b}}_t = \exp\left\{\int_0^t du\, y^{\mathrm{b}}_u\right\}$$

- We define also a TBA rate as given by

$$f_t := 1_{\{q^f_t>0\}}y^{\mathrm{b}}_u + 1_{\{q^f_t\leq 0\}}\left(y^{\mathrm{l}}_u - s^{\mathrm{l}}_u\right)$$

Trading Strategies with Funding Costs – III

- To avoid arbitrages we require that the gain processes, deflated by the TBA, are martingales under the $\mathcal{G}$ filtration.
  - ⟶ The equivalent martingale measure depends on the netting set, so that we are removing only the arbitrages within the netting set.
  - ⟶ See Bielecki and Rutkowski (2014) for a discussion of arbitrages in non-linear pricing.
- The pricing equation becomes

$$
\begin{aligned}
1_{\{\tau>t\}}\widetilde{S}_t \;&=\; B^f_t\,\mathbb{E}^f\left[1_{\{t<\tau\leq T\}}\frac{\theta_\tau}{B^f_\tau} + 1_{\{t<\tau=\tau_I\leq T\}}(q^f_{\tau_I})^+(1-R_I) \,\middle|\, \mathcal{G}_t \right] \\
&+\; B^f_t\,\mathbb{E}^f\left[\int_t^T 1_{\{\tau>u\}}\left(\frac{d\pi_u}{B^f_u} + \frac{(f_u-c_u)C_u\,du}{B^f_u} \right) \,\middle|\, \mathcal{G}_t \right]
\end{aligned}
$$

where thw f over the expectation symbols is reminder of the dependency of the measure on the funding strategy.

Funding Policies – I

- A first possibility to fix the unknown value of $(q^f_{\tau_I})^+$ is pricing the whole netting set by replicating it with the contained securities.
- This price-and-hedge problem requires to simultaneously solve three equations: the definition of the netting set, the self-financing condition, and the terminal condition inclusive of the funding benefit.
- The solution consists both in the value of the netting set and in the strategy in cash and securities used to hedge it.
  $\longrightarrow$ See Crépey (2011), Bielecki and Rutkowski (2014).
- The existence of a solution in a general setting is difficult to prove, since the terminal condition is not predictable (gap risk).
  $\longrightarrow$ Contagion effects, or a delay in the default procedure, cannot be hedged in the practice.
  $\longrightarrow$ The market securities may jump if sensitive to credit risk.

Funding Policies – II

- Alternatively, we could implement a partial hedging, and we could size the cash amount by some optimal argument.
  - $\longrightarrow$ See Crépey (2011), Burgard and Kjaer (2011).
- Here, we assume a diffusive setting for underlying risk factors and predictable terminal conditions for price processes.
  - $\longrightarrow$ Under these assumptions, we know that, in case of a complete market, the hedging strategy is given by delta hedging.
  - $\longrightarrow$ See Crépey (2011), Pallavicini, Perini and Brigo (2011).
- Thus, if we can consider a netting set formed by one derivative security V_t along with its delta-hedging assets S_t, we get that the quantity of cash needed to implement the hedging strategy is given by

$$F_t := V_t - C_t^V - (S_t - C_t) \cdot \partial_S V_t \ , \quad q_t^f \doteq \frac{F_t + \epsilon_t}{B_t^f}$$

where C_t and C_t^V are the collateral accounts of market and derived securities, and ϵ_t is the hedging error.

Funding Policies – III

- In the same way we can consider netting sets formed by many derived securities. In this case, we can net all the funding requirements.

$$F_t^i := V_t^i - C_t^V - (S_t - C_t) \cdot \partial_S V_t^i \,, \quad q_t^f \doteq \frac{1}{B_t^f} \sum_{i=1}^{n} (F_t^i + \epsilon_t^i)$$

- Notice that this choice effectively reduces funding requirements, since we have

$$\left(\sum_{i=1}^{n} (F_t^i + \epsilon_t^i) \right)^+ \leq \sum_{i=1}^{n} \left(F_t^i + \epsilon_t^i\right)^+$$

- In the following we focus on the case of a single derived security. In the last section we discuss again this problem.

Pricing Formulae with Funding Costs – I

- In the case of a netting set formed by a single derived security, and under the assumption of a diffusive setting without gap risk, we can write the price equation as given by

$$
\begin{aligned}
1_{\{\tau>t\}} \widetilde{V}_t &= B^f_t \, \mathbb{E}^f \left[1_{\{t<\tau\leq T\}} \frac{\theta^V_\tau}{B^f_\tau} + \int_t^T 1_{\{\tau>u\}} \frac{d\pi^V_u}{B^f_u} + \frac{(f_u - c_u) C^V_u \, du}{B^f_u} \,|\, \mathcal{G}_t \right] \\
&+ B^f_t \, \mathbb{E}^f \left[1_{\{t<\tau=\tau_I\leq T\}} (1 - R_I) \frac{(F_{\tau_I} + \epsilon_{\tau_I})^+}{B^f_{\tau_I}} \,|\, \mathcal{G}_t \right]
\end{aligned}
$$

- It is useful to make explicit the dependency of the TBA process and rate on the cash used to implement the hedging strategy.

Pricing Formulae with Funding Costs – II

- We can apply the Feynman-Kac theorem to write flows deflated w.r.t. a bank account B^e_t accruing at the overnight rate e_t.

$$
\begin{aligned}
1_{\{\tau>t\}}\widetilde{V}_t &= B^e_t \int_t^T \mathbb{E}^e\left[1_{\{\tau>u\}} \frac{d\pi^V_u + (e_u - c_u) C^V_u \, du}{B^e_u} + 1_{\{\tau\in du\}} \frac{\theta^V_u}{B^e_u} \,|\, \mathcal{G}_t \right] \\
&- B^e_t \int_t^T \mathbb{E}^e\left[1_{\{\tau>u\}} (f_u - e_u) \frac{F_u}{B^e_u} \,|\, \mathcal{G}_t \right] \\
&+ B^e_t \int_t^T \mathbb{E}^e\left[1_{\{\tau=\tau_I\in du\}} (1 - R_I) \frac{(F_u + \epsilon_u)^+}{B^e_u} \,|\, \mathcal{G}_t \right]
\end{aligned}
$$

where the pricing measure is such that the market securities can be priced as

$$
1_{\{\tau>t\}}\widetilde{S}_t = B^e_t \int_t^T \mathbb{E}^e\left[1_{\{\tau>u\}} \frac{d\pi_u + (e_u - c_u) C_u \, du}{B^e_u} + 1_{\{\tau\in du\}} \frac{\theta_u}{B^e_u} \,|\, \mathcal{G}_t \right]
$$

Does Funding Costs Exists ? – I

- We can check that in a complete market funding costs does not exits.
  $\longrightarrow$ See Pallavicini, Perini and Brigo (2011), Burgard and Kjaer (2011), Hull and White (2012), Albanese and Andersen (2015).
- We switch to market filtration, remove collaterals, and we put $\epsilon_t \doteq 0$.

$$
\begin{aligned}
1_{\{\tau>t\}}\widetilde{V}_t \;=\;& 1_{\{\tau>t\}}\int_t^T \mathbb{E}^e\left[\frac{B_t^{e+\lambda}}{B_u^{e+\lambda}}\left(d\pi_u^V + \theta_u^V\lambda_u\,du\right)\,|\,\mathcal{F}_t\right] \\
&- 1_{\{\tau>t\}}\int_t^T du\,\mathbb{E}^e\left[\frac{B_t^{e+\lambda}}{B_u^{e+\lambda}}\ell_u^{\mathrm{l}}\left(V_u - S_u\cdot\partial_S V_u\right)^-\,|\,\mathcal{F}_t\right] \\
&- 1_{\{\tau>t\}}\int_t^T du\,\mathbb{E}^e\left[\frac{B_t^{e+\lambda}}{B_u^{e+\lambda}}(\ell_u^{\mathrm{b}} + s_u^{\mathrm{b}} - (1-R_I)\lambda_u^I)\left(V_u - S_u\cdot\partial_S V_u\right)^+\,|\,\mathcal{F}_t\right]
\end{aligned}
$$

where we have substituted the TBA rate in term of CDS spreads and liquidity bases, while the default intensities are defined as

$$
\lambda_t\,dt := \mathbb{E}\left[1_{\{\tau\in t\}}\,|\,\mathcal{G}_t\right]\;,\quad \lambda_t^I\,dt := \mathbb{E}\left[1_{\{\tau_I\in t\}}\,|\,\mathcal{G}_t\right]
$$

Does Funding Costs Exists ? – II

- Then, we apply again the Feynman-Kac theorem to group all the adjustments into an effective discount rate.

$$1_{\{\tau>t\}}\widetilde{V}_t = 1_{\{\tau>t\}}\int_t^T \mathbb{E}^\zeta\left[\frac{B_t^{\zeta+\lambda}}{B_u^{\zeta+\lambda}}\left(d\pi_u^V + \theta_u^V\lambda_u\,du\right) \mid \mathcal{F}_t\right]$$

where the effective rate ζ_t is given by

$$\zeta_t := e_t + (\ell_t^{\mathrm{b}} + s_t^{\mathrm{b}} - (1-R_I)\lambda_t^I)1_{\{V_t>S_t\cdot\partial_S V_t\}} + \ell_t^{\mathrm{l}}1_{\{V_t<S_t\cdot\partial_S V_t\}}$$

- If the bank has the possibility to trade her own CDS, we have that

$$s_t^{\mathrm{b}} \doteq (1-R_I)\lambda_t^I$$

leading to

$$\zeta_t = e_t + \ell_t^{\mathrm{b}}1_{\{V_t>S_t\cdot\partial_S V_t\}} + \ell_t^{\mathrm{l}}1_{\{V_t<S_t\cdot\partial_S V_t\}} \doteq r_t$$

where the last step holds assuming no CDS/bond basis.

Fair Value Policies – I

- Some cash flows in the pricing equation happen after the default of the investor.
  $\longrightarrow$ These flows are terms in θ_t^V and the funding benefits.
- The investor can ignore such flows while trading with other counterparties, since they matter only when the default procedure is in place after the investor default.
- We can split accordingly the derivative price as

$$V_t := V_t^1 + V_t^2$$

  where V_t^1 is the trading part and V_t^2 the treasury part of the derivative price.
- Furthermore, we split the on-default cash flow to isolate the part occurring on investor's default.

$$\theta_\tau^V := 1_{\{\tau_C < \tau_I\}} \theta_{\tau_C}^{V,C} + 1_{\{\tau_I < \tau_C\}} \theta_{\tau_I}^{V,I}$$

Fair Value Policies – II

- We start with an approximation. We consider the funding adjustments only for contractual cash flows, and we discard the hedging error, so that we get

$$
\begin{aligned}
V_t \quad \approx & \quad \int_t^T \mathbb{E}^e\left[1_{\{\tau > u\}} \frac{B_t^e}{B_u^e} \left(d\pi_u^V + (e_u - c_u) C_u^V \, du + 1_{\{\tau \in du\}} \theta_u^V \right) \,|\, \mathcal{G}_t \right] \\
+ & \quad \int_t^T \mathbb{E}^e\left[1_{\{\tau > u\}} \frac{B_t^e}{B_u^e} \left(1_{\{\tau_I \in du\}} (1 - R_I)(F_u^0)^+ - (f_u^0 - e_u) F_u^0 \, du \right) \,|\, \mathcal{G}_t \right]
\end{aligned}
$$

$$
F_t^0 := V_t^0 - C_t^V - (S_t - C_t) \cdot \partial_S V_t^0 \,, \quad V_t^0 := \int_t^T \mathbb{E}_t^e\left[\frac{B_t^e}{B_u^e} \, d\pi_u \right]
$$

$$
f_t^0 := e_t + 1_{\{F_t^0 > 0\}} \left(s_t^{\mathrm{b}} + \ell_t^{\mathrm{b}} \right) + 1_{\{F_t^0 \leq 0\}} \ell_t^{\mathrm{l}}
$$

- Notice that if the market is complete this approximation is exact.

Fair Value Policies – III

MtM	$V_t^1 := V_t^0$
CVA	$- \int_t^T \mathbb{E}^e \left[1_{\{\tau=\tau_C \in du\}} \frac{B_t^e}{B_u^e} (V_u^0 - \theta_u^C) \mid \mathcal{G}_t \right]$
LVA	$- \int_t^T \mathbb{E}^e \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} (c_u - e_u) C_u^V \, du \mid \mathcal{G}_t \right]$
FCA	$- \int_t^T \mathbb{E}^e \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \left(s_u^{\mathrm{b}} + \ell_u^{\mathrm{b}}\right) (F_u^0)^+ \, du \mid \mathcal{G}_t \right]$
FBA	$+ \int_t^T \mathbb{E}^e \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \ell_t^{\mathrm{l}} (-F_u^0)^+ \, du \mid \mathcal{G}_t \right]$
DVA	$V_t^2 := \int_t^T \mathbb{E}^e \left[1_{\{\tau=\tau_I \in du\}} \frac{B_t^e}{B_u^e} (\theta_u^I - V_u^0) \mid \mathcal{G}_t \right]$
FDA	$+ \int_t^T \mathbb{E}^e \left[1_{\{\tau=\tau_I \in du\}} \frac{B_t^e}{B_u^e} (1 - R_I)(F_u^0)^+ \mid \mathcal{G}_t \right]$

Fair Value Policies on Netting Sets – I

- If we look at the whole netting set, we can apply the previous decomposition to each contract of the set, but with a cash amount

$$F_t^0 := \sum_{i=1}^{n} F_t^{0,i} := \sum_{i=1}^{n} V_t^{0,i} - C_t^{V,i} - (S_t - C_t) \cdot \partial_S V_t^{0,i}$$

- Since the adjustments are non-linear functions of the cash amount, we need a recipe to decompose the adjustments.
- We can define the cash amount to compute funding costs

$$F_t^{\mathrm{b},0,i} := (\sum_{j=1}^{n} F_t^{0,j})^+ - (\sum_{j=1,j\neq i}^{n} F_t^{0,j})^+$$

and the cash amount to compute funding benefits

$$F_t^{\mathrm{l},0,i} := (\sum_{j=1}^{n} F_t^{0,j})^- - (\sum_{j=1,j\neq i}^{n} F_t^{0,j})^-$$

Fair Value Policies on Netting Sets – II

- Thanks to the definitions of the cash amounts to compute funding costs and benefits, we can add a new contract into the netting set without re-computing the funding adjustments of the other ones.

$$
\begin{aligned}
\text{FCA}^i \quad & - \int_t^T \mathbb{E}^e \left[1_{\{\tau > u\}} \frac{B_t^e}{B_u^e} \left(s_u^{\mathrm{b}} + \ell_u^{\mathrm{b}} \right) F_u^{\mathrm{b},0,i} \, du \mid \mathcal{G}_t \right] \\
\text{FBA}^i \quad & + \int_t^T \mathbb{E}^e \left[1_{\{\tau > u\}} \frac{B_t^e}{B_u^e} \ell_t^{\mathrm{l}} (-F_u^{\mathrm{l},0,i}) \, du \mid \mathcal{G}_t \right] \\
\text{FDA}^i \quad & + \int_t^T \mathbb{E}^e \left[1_{\{\tau = \tau_I \in du\}} \frac{B_t^e}{B_u^e} (1 - R_I) F_u^{\mathrm{l},0,i} \mid \mathcal{G}_t \right]
\end{aligned}
$$

Probabilistic Interpretation of Pricing Equations – I

The Feynman-Kac Theorem

Consider a vector of Markov risk factors S_t with infinitesimal generator

$$\mathcal{L}_t^{\mu} := (\mu_t S_t) \cdot \partial_S + \frac{1}{2}\mathrm{Tr}\partial_t \langle S, S\rangle_t \partial_S^2$$

and assume that the derivative price V_t solves the PDE

$$(\partial_t + \mathcal{L}_t^{\mu} - \nu_t)\, V_t + \partial_t \pi_t = 0 \,, \quad V_T = 0$$

Hence, the solution of the PDE is given by

$$V_t = \int_t^T \mathbb{E}_t^{\mu}\left[\frac{B_t^{\nu}}{B_u^{\nu}}\, d\pi_u \right]$$

where under the pricing measure $\mathbb{Q}^{\mu}$ the risk factors grow at rate μ_t.

Probabilistic Interpretation of Pricing Equations – II

- A useful application of the theorem is changing the discount factor by adding a stream of coupons.

$$
\begin{aligned}
V_t &= \int_t^T \mathbb{E}_t^{\mu}\left[\frac{B_t^{\nu}}{B_u^{\nu}} \, d\pi_u \right] \\
&= \int_t^T \mathbb{E}_t^{\mu}\left[\frac{B_t^{\rho}}{B_u^{\rho}} \, d\pi_u + (\mu_u - \rho_u) V_u \, du \right] \\
&= \int_t^T \mathbb{E}_t^{\rho}\left[\frac{B_t^{\rho}}{B_u^{\rho}} \, d\pi_u + (\mu_u - \rho_u) V_u \, du - (\nu_u - \rho_u) S_u \cdot \partial_S V_u \, du \right]
\end{aligned}
$$

where under the pricing measure $\mathbb{Q}^{\rho}$ the risk factors grow at rate ρ_t.

Pricing Cash Flows Occurring before the Default Event – I

- For any $\mathcal{G}$-adapted process ϕ_t, we can consider the $\mathcal{G}$-adapted process

$$x_t \doteq \mathbb{E}\left[\,1_{\{\tau>T\}}\phi_T \mid \mathcal{G}_t\,\right]$$

- If we observe x_t only before the default event, and we take the expectations of both side under $\mathcal{F}$ filtration, we get

$$\tilde{x}_t\,\mathbb{E}\left[\,1_{\{\tau>t\}} \mid \mathcal{F}_t\,\right] = \mathbb{E}\left[\,1_{\{\tau>t\}}\mathbb{E}\left[\,1_{\{\tau>T\}}\phi_T \mid \mathcal{G}_t\,\right] \mid \mathcal{F}_t\,\right] = \mathbb{E}\left[\,1_{\{\tau>T\}}\phi_T \mid \mathcal{F}_t\,\right]$$

On the other hand, we have from the definition of pre-default process

$$1_{\{\tau>t\}}\tilde{x}_t = 1_{\{\tau>t\}}\mathbb{E}\left[\,1_{\{\tau>T\}}\phi_T \mid \mathcal{G}_t\,\right]$$

leading to

$$1_{\{\tau>t\}}\mathbb{E}\left[\,1_{\{\tau>T\}}\phi_T \mid \mathcal{G}_t\,\right] = 1_{\{\tau>t\}}\frac{\mathbb{E}\left[\,1_{\{\tau>T\}}\phi_T \mid \mathcal{F}_t\,\right]}{\mathbb{Q}\{\,\tau>t \mid \mathcal{F}_t\,\}}$$

Pricing Cash Flows Occurring before the Default Event – II

First Filtration Switching Lemma

In a market with defaultable names, where τ is the first default event, we can price cash flows occurring before the first default event by switching to the market filtration $\mathcal{F}$.

$$1_{\{\tau>t\}}\mathbb{E}\left[\, 1_{\{\tau>T\}}\phi_T \,|\, \mathcal{G}_t \right] = 1_{\{\tau>t\}}\frac{\mathbb{E}\left[\, \mathbb{Q}\{\, \tau > T \,|\, \mathcal{F}_T \,\}\, \tilde{\phi}_T \,|\, \mathcal{F}_t \,\right]}{\mathbb{Q}\{\, \tau > t \,|\, \mathcal{F}_t \,\}}$$

where ϕ_t is a $\mathcal{G}$-adapted process, and ϕx_t is the corresponding pre-default process. In particular, we have also

$$1_{\{\tau>t\}}\mathbb{Q}\{\, \tau > T \,|\, \mathcal{G}_t \,\} = 1_{\{\tau>t\}}\frac{\mathbb{Q}\{\, \tau > T \,|\, \mathcal{F}_t \,\}}{\mathbb{Q}\{\, \tau > t \,|\, \mathcal{F}_t \,\}}$$

Pricing Cash Flows Occurring on the Default Event – I

- A second useful lemma can be derived for cash flows paid only if a default occurs.
- For any $\mathcal{G}$-adapted process ϕ_t we can proceed as before, but, now, we consider the $\mathcal{G}$-adapted process

$$x_t \doteq \mathbb{E}\left[\, 1_{\{\tau<T\}} \phi_\tau \,|\, \mathcal{G}_t \,\right]$$

leading to

$$1_{\{\tau>t\}} \mathbb{E}\left[\, 1_{\{\tau<T\}} \phi_\tau \,|\, \mathcal{G}_t \,\right] = 1_{\{\tau>t\}} \frac{\mathbb{E}\left[\, 1_{\{t<\tau<T\}} \phi_\tau \,|\, \mathcal{F}_t \,\right]}{\mathbb{Q}\{\, \tau > t \,|\, \mathcal{F}_t \,\}}$$

- As before we wish to remove the explicit dependency on the default event on the right-hand side.

Pricing Cash Flows Occurring on the Default Event – II

- We go on by localizing the default event, and we get

$$1_{\{\tau>t\}}\mathbb{E}\left[\, 1_{\{t<\tau<T\}}\phi_\tau \,\middle|\, \mathcal{F}_t \right] = 1_{\{\tau>t\}} \int_t^T \mathbb{E}\left[\, 1_{\{\tau\in du\}}\phi_u \,\middle|\, \mathcal{F}_t \right]$$

- To proceed further we require that ϕ_t is also predictable. We obtain

$$1_{\{\tau>t\}}\mathbb{E}\left[\, 1_{\{t<\tau<T\}}\phi_\tau \,\middle|\, \mathcal{F}_t \right] = 1_{\{\tau>t\}} \int_t^T du\, \mathbb{E}\left[\, 1_{\{\tau>u\}}\lambda_u\phi_u \,\middle|\, \mathcal{F}_t \right]$$

where we define the first-default intensity as the density of the compensator of $1_{\{\tau<t\}}$, namely

$$\lambda_t\, dt := \mathbb{E}\left[\, 1_{\{\tau\in dt\}} \,\middle|\, \mathcal{G}_t \right]$$

Pricing Cash Flows Occurring on the Default Event – III

Second Filtration Switching Lemma

In a market with defaultable names, where τ is the first default event, we can price cash flows occurring on the first default event by switching to the market filtration $\mathcal{F}$.

$$1_{\{\tau>t\}}\mathbb{E}\left[\,1_{\{\tau<T\}}\phi_T \,|\, \mathcal{G}_t\,\right] = 1_{\{\tau>t\}} \int_t^T du\, \frac{\mathbb{E}\left[\,\mathbb{Q}\{\,\tau > u \,|\, \mathcal{F}_u\,\}\,\tilde{\lambda}_u \tilde{\phi}_u \,|\, \mathcal{F}_t\,\right]}{\mathbb{Q}\{\,\tau > t \,|\, \mathcal{F}_t\,\}}$$

where λ_t is the first-default intensity and ϕ_t is a $\mathcal{G}$-predictable process, while $\tilde{\lambda}_t$ and $\tilde{\phi}_t$ are the corresponding pre-default processes.

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