Diamond trees and the forest expansion

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Outline of this talk

- The diamond product
- The $\mathbb{G}$-expansion
  - Trees and forests
- The $\mathbb{K}$-expansion
  - Third cumulant
- The $\mathbb{F}$-expansion and stochastic volatility
  - Triple joint MGF
  - The leverage swap
  - The Bergomi-Guyon smile expansion to all orders
- Explicit computations in affine forward variance models

This work and the exponentiation theorem

- In earlier talks, I presented the diamond product and the exponentiation theorem.
- Manipulations were formal and the convergence properties of the resulting forest expansion unclear.
  - Eventually published as Elisa Alòs, Jim Gatheral, and Radoš Radoičić, Exponentiation of conditional expectations under stochastic volatility, *Quantitative Finance* 20(1):13–27, 2020.
- This time I explain the remarkably simple origin of the forest expansion, I give its convergence properties and attempt to give a sense of its wide applicability.

The diamond product

Definition

Given two continuous semimartingales A, B with integrable covariation process $\langle A, B \rangle$, the diamond product[a] of A and B is another continuous semimartingale given by

$$(A \diamond B)_t(T) := \mathbb{E}_t \left[\langle A, B \rangle_{t,T} \right] = \mathbb{E}_t \left[\langle A, B \rangle_T \right] - \langle A, B \rangle_t ,$$

where $\langle A, B \rangle_{t,T} = \langle A, B \rangle_T - \langle A, B \rangle_t$.

[a] Warning. Our diamond product is (very) different from the Wick product.

Properties of the diamond product

- Commutative: $A \diamond B = B \diamond A$.
- Non-associative: $(A \diamond B) \diamond C \neq A \diamond (B \diamond C)$.
- $A \diamond B$ depends only on the respective martingale parts of A and B.
- $A \diamond B$ is in general not a martingale.

The $\mathbb{G}$-forest expansion

Theorem 1 (Theorem 1.1 of [FGR20])

Let Y_T be a real-valued, $\mathcal{F}_T$-measurable random variable with associated martingale $Y_t = \mathbb{E}_t\left[Y_T\right]$. Under natural integrability conditions, with a, b small enough, there is a.s. convergence of

$$\log \mathbb{E}_t\left[e^{aY_T + b\langle Y\rangle_T}\right] = aY_t + b\langle Y\rangle_t + \sum_{k\geq 2} \mathbb{G}_t^k(T), \tag{1}$$

where

$$\begin{aligned} \mathbb{G}^2 &= \left(\frac{1}{2}a^2 + b\right)(Y \diamond Y)_t(T), \\ \mathbb{G}^k &= \frac{1}{2}\sum_{j=2}^{k-2} \mathbb{G}^{k-j} \diamond \mathbb{G}^j + \left(a\, Y \diamond \mathbb{G}^{k-1}\right) \text{ for } k > 2. \end{aligned} \tag{2}$$

Idea of the proof

For a generic (continuous) semimartingale Z, sufficiently integrable, let

$$\Lambda_t^T = \log \mathbb{E}_t \left[e^{Z_{t,T}} \right].$$

Then, noting that $\Lambda_T^T = 0$,

$$\mathbb{E}_t \left[e^{Z_T} \right] = \mathbb{E}_t \left[e^{Z_T + \Lambda_T^T} \right] = e^{Z_t + \Lambda_t^T}.$$

The stochastic logarithm $\mathcal{L}\left(\mathbb{E}_\bullet(Z_T)\right) = Z + \Lambda^T + \frac{1}{2}\langle Z + \Lambda^T \rangle$ is a martingale. Thus,

$$\begin{aligned} \Lambda_t^T &= \mathbb{E}_t \left[Z_{t,T} + \tfrac{1}{2}\langle Z + \Lambda^T \rangle_{t,T} \right] \\ &= \mathbb{E}_t \left[Z_{t,T} \right] + \tfrac{1}{2}((Z + \Lambda^T) \diamond (Z + \Lambda^T))_t(T). \end{aligned}$$

Now with[1] $Z = \epsilon a Y + \epsilon^2\, b\langle Y\rangle$ we get

$$\Lambda_t^T(\epsilon) = \epsilon a\, \mathbb{E}_t\left[Y_{t,T}\right] + \epsilon^2\, b\,(Y \diamond Y)_t(T) + \tfrac{1}{2}\left(\epsilon a Y + \Lambda_t^T(\epsilon)\right)_t^{\diamond 2}(T)\,.$$

Put $\Lambda_t^T(\epsilon) = \epsilon^2 \mathbb{G}_t^2 + \epsilon^3 \mathbb{G}_t^3 + \ldots$, and match coefficients of ϵ^n.

$[\epsilon^2]$: $\mathbb{G}_t^2 = b\,(Y \diamond Y)_t(T) + \frac{1}{2} a^2\,(Y \diamond Y)_t(T)$.

$[\epsilon^3]$: $\mathbb{G}_t^3 = (a\, Y \diamond \mathbb{G}^2)_t(T)$.

$[\epsilon^4]$: $\mathbb{G}_t^4 = (a\, Y \diamond \mathbb{G}^3)_t(T) + \frac{1}{2}(\mathbb{G}^2 \diamond \mathbb{G}^2)_t(T)$.

- We see the recursion (2) emerge!

[1] Recall that terms of bounded variation such as $\langle Y\rangle$ do not contribute to diamond products.

Special cases

Interesting special cases include

- The exponential martingale: $b = -\frac{1}{2}a^2$. All corrector terms $\mathbb{G}^k$ vanish.
  - The $\mathbb{G}$-expansion can thus be seen as a "broken exponential martingale" expansion.
- The $\mathbb{F}$-forest expansion of [AGR2020] (working paper 2017): $\frac{1}{2}a + b = 0$.
  - The $\mathbb{F}$-forest expansion gives a general expression for the characteristic function of the log-stock price in a stochastic volatility model written in forward variance form.
- The cumulant ($\mathbb{K}$-forest) expansion of Lacoin-Rhodes-Vargas [LRV19]: $b = 0$.
  - Their expansion was derived in the context of renormalization of the sine-Gordon model in quantum physics.

Further applications

- In [FGR20], we give a number of applications.
- Other possible applications include
  - computation of likelihood functions in statistics,
  - computation of correlation functions in statistical physics,
  - computation of amplitudes in quantum field theory.

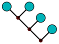
- It's very satisfying that problems in quantitative finance and quantum physics lead to the same nice mathematics!

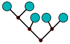
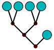
Trees and forests

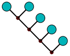
- The general term $\mathbb{G}^n_t(T)$ in (2) is naturally written as a linear combination of binary diamond trees[2].
- Hence the terminology $\mathbb{G}$-*forest expansion* for (1).
- Specifically, writing • as a short-hand for Y, interpreted as single leaf, we have

$\mathbb{G}^2 = \left(\frac{1}{2}a^2 + b\right)$ 

$\mathbb{G}^3 = a\left(\frac{1}{2}a^2 + b\right)$ 

$\mathbb{G}^4 = \frac{1}{2}\left(\frac{1}{2}a^2 + b\right)^2$  $+\, a^2\left(\frac{1}{2}a^2 + b\right)$ 

$\mathbb{G}^5 = a\left(\frac{1}{2}a^2 + b\right)^2$  $+\, \frac{1}{2}a\left(\frac{1}{2}a^2 + b\right)^2$ 

$+\, a^3\left(\frac{1}{2}a^2 + b\right)$  (3)

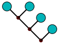
[2] Trees stolen from [Hai13]!

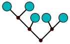
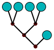
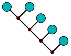
The $\mathbb{K}$-forest expansion

As mentioned earlier, the $\mathbb{K}$-forest expansion ($\mathbb{K}$ for "Kumulant") is obtained by setting $b = 0$ in (1). This gives

$\mathbb{K}^2 = \frac{1}{2} a^2$ 

$\mathbb{K}^3 = \frac{1}{2} a^3$ 

$\mathbb{K}^4 = \frac{1}{8} a^4$  $+ \frac{1}{2} a^4$ 

$\mathbb{K}^5 = \frac{1}{4} a^5$  $+ \frac{1}{8} a^5$  $+ \frac{1}{2} a^5$ 

With $\mathbb{K}^1 = \bullet$, the $\mathbb{K}$-recursion follows naturally.

The $\mathbb{K}$-forest expansion

Theorem 2 (Theorem 1.2 of [FGR20])

Let A_T be $\mathcal{F}_T$-measurable with $N \in \mathbb{N}$ finite moments. Then the recursion

$$\mathbb{K}_t^{n+1}(T) = \frac{1}{2}\sum_{k=1}^{n}(\mathbb{K}^k \diamond \mathbb{K}^{n+1-k})_t(T), \quad \forall n > 0$$

with $\mathbb{K}_t^1(T) := \mathbb{E}_t\left[A_T\right]$ is well-defined up to $\mathbb{K}^N$ and, for $a \in \mathbb{R}$,

$$\log \mathbb{E}_t\left[e^{iaA_T}\right] = \sum_{n=1}^{N}(ia)^n\,\mathbb{K}_t^n(T) + o(|a|^N)$$

which identifies $n! \times \mathbb{K}_t^n(T)$ as the (time t-conditional) n.th cumulant of A_T.

Example: $\mathbb{K}^3$ and the third central moment

- For higher n, the forest expansion encodes relations that are increasingly complex to derive by hand.
- For example, from the forest expansion we have

$$\mathbb{K}^3_t(T) = \tfrac{1}{2}\left(Y \diamond (Y \diamond Y)\right)_t(T)$$

and also, since the third cumulant is the third central moment,

$$\mathbb{K}^3_t(T) = \frac{1}{3!}\,\mathbb{E}_t\left[{Y_{t,T}}^3\right].$$

- On the other hand, the relation

$$\tfrac{1}{2}\left(Y \diamond (Y \diamond Y)\right)_t(T) = \frac{1}{3!}\,\mathbb{E}_t\left[{Y_{t,T}}^3\right]$$

is not so obvious.

Another application: MGF of the Lévy area

Theorem (P. Lévy)

Let $\{X, Y\}$ be 2-dimensional standard Brownian motion, and stochastic ("Lévy") area be given by

$$\mathcal{A}_t = \int_0^t (X_s\, dY_s - Y_s\, dX_s) \ .$$

Then, for $T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\mathbb{E}_0\left[e^{\mathcal{A}_T}\right] = \frac{1}{\cos T} \ .$$

- In particular, we will see how to compute trees in practice.

First term

First,

$$\mathbb{K}^2 = \frac{1}{2}$$ 

$$\begin{aligned} &= \frac{1}{2}(\mathcal{A} \diamond \mathcal{A})_t(T) \\ &= \frac{1}{2}\int_t^T \left(\mathbb{E}_t\left[X_s^2\right] + \mathbb{E}_t\left[Y_s^2\right]\right)\, ds \\ &= \frac{1}{2}(T-t)^2 + \frac{1}{2}(X_t^2 + Y_t^2)(T-t). \end{aligned}$$

In particular,

$$d\mathbb{K}_s^2 = (X_s\, dX_s + Y_s\, dY_s)(T-s) + \mathrm{BV},$$

where BV denotes a bounded variation term.

- Note that BV terms do not contribute to diamond trees.

Second term

Similarly, recalling that $d\mathbb{K}^1_s = X_s\, dY_s - Y_s\, dX_s$,

$$\mathbb{K}^3 = \mathbb{K}^1 \diamond \mathbb{K}^2 =$$ 

$$= \mathbb{E}_t\left[\int_t^T d\langle \mathbb{K}^1, \mathbb{K}^2\rangle_s\right]$$

$$= \mathbb{E}_t\left[\int_t^T \left[XY\, d\langle Y\rangle_s - YX\, d\langle X\rangle_s\right]\,(T-s)\right] = 0.$$

- It is easy to check that all odd forests vanish.

$\mathbb{K}^4$

$$\mathbb{K}^4 = \frac{1}{2}\,\mathbb{K}^2 \diamond \mathbb{K}^2 = \frac{1}{2}$$

$$\begin{aligned}
&= \frac{1}{2}\,\mathbb{E}_t\left[\int_t^T \left[X_s^2\, d\langle X\rangle_s + Y_s^2\, d\langle Y\rangle_s\right](T-s)^2\right] \\
&= \frac{1}{2}\int_t^T \left(\mathbb{E}_t\left[X_s^2\right] + \mathbb{E}_t\left[Y_s^2\right]\right)(T-s)^2\, ds \\
&= \int_t^T (s-t)\,(T-s)^2\, ds + \frac{1}{2}\,(X_t^2 + Y_t^2)\int_t^T (T-s)^2\, ds \\
&= \frac{1}{12}(T-t)^4 + \frac{1}{2}\,(X_t^2 + Y_t^2)\,\frac{1}{3}\,(T-t)^3.
\end{aligned}$$

- It is now clear how to extend this computation to all orders.

The general pattern

We see that for each even n, $\mathbb{K}_t^n(T) = a_n\, I_t^{(n)}(T)$ for some $a_n \in \mathbb{Q}$ where

$$\begin{aligned} I_t^{(n)}(T) &= \frac{1}{2}\int_t^T \left(\mathbb{E}_t\left[X_s^2\right] + \mathbb{E}_t\left[Y_s^2\right]\right)(T-s)^{n-2}\, ds \\ &= \frac{(T-t)^n}{n(n-1)} + \frac{1}{2}\left(X_t^2 + Y_t^2\right)\frac{1}{n-1}(T-t)^{n-1}. \end{aligned}$$

To compute the forests $\mathbb{K}^n$, we need the following lemma.

Lemma

$$\left(I^{(m)} \diamond I^{(n)}\right)_t(T) = \frac{2}{(m-1)(n-1)}\, I_t^{(n+m)}(T).$$

More terms

- Note from above that $\mathbb{K}^2 = I^{(2)}$ and $\mathbb{K}^4 = I^{(4)}$.
- Applying the lemma

$$\begin{aligned} \mathbb{K}^6 &= I^{(4)} \diamond I^{(2)} = \frac{2}{3 \cdot 1} I^{(6)} \\ &= \frac{(T-t)^6}{45} + \frac{2}{3}\frac{1}{2}(X_t^2 + Y_t^2)\frac{1}{5}(T-t)^5. \end{aligned}$$

- In principle, we could go on for ever, computing forests (or cumulants) in this way.
  - As we show in [FGR20], without much extra effort, we can sum all these cumulants and so recover Lévy's theorem.

Remark

As a comparison, Levin and Wildon[LW08] obtain Lévy's theorem from (a much harder) moment expansion.

A bivariate $\mathbb{K}$-expansion

Let $\mathbb{K}^1_t = \mathbb{E}_t \left[a \, Y_T + b \, \langle Y \rangle_{t,T} \right] \equiv a + b$. Then

$$
\begin{aligned}
\mathbb{K}^1 &= a + b \\
\mathbb{K}^2 &= \tfrac{1}{2} \left(a + b \right)^{\diamond 2} = \tfrac{1}{2} a^2 + ab + \tfrac{1}{2} b^2 \\
\mathbb{K}^3 &= \tfrac{1}{2} a^3 + \tfrac{1}{2} a^2 b + a^2 b + ab^2 + \tfrac{1}{2} ab^2 + \ldots \\
\mathbb{K}^4 &= \tfrac{1}{2} a^4 + \tfrac{1}{2^3} a^4 + \tfrac{1}{2} a^3 b + \tfrac{1}{2} a^3 b \\
&\quad + a^3 b + \tfrac{1}{2} a^3 b + \ldots \\
\mathbb{K}^5 &= \tfrac{1}{2} a^5 + \tfrac{1}{2^3} a^5 + \tfrac{1}{2^2} a^5 + \ldots \qquad (4)
\end{aligned}
$$

Forest reordering

- We see that the $\mathbb{G}$-recursion is equivalent to the bivariate $\mathbb{K}$-recursion applied to $A_T = aY_T + b\,\langle Y\rangle_T$, after forest reordering.
  - Reorder by collecting all trees with the same number of leaves.
  - $\mathbb{G}$-forests consist of trees which are homogenous in the number of leaves $\bullet$ but not in a, b.
- Note also that forest reordering resolves the infinite cancellations present in the bivariate $\mathbb{K}$-expansion.
  - To see this put $b = -\frac{1}{2}a^2$ in (4) – we see a very complicated expression which must sum to zero.
  - On the other hand putting $b = -\frac{1}{2}a^2$ in (3) trivially results in zero.

Forward variance models

- Let S be a strictly positive continuous martingale.
- Then $X := \log S$ is a semimartingale with quadratic variation process $\langle X \rangle$.
- Following [BG12], it is natural to specify a model in forward variance form.

$$\begin{aligned} v_t\, dt &:= d\langle X \rangle_t \\ \xi_t(T) &= \mathbb{E}_t \left[v_T \right]. \end{aligned}$$

  - Forward variances are tradable assets (unlike spot variance).
  - We get a family of martingales indexed by their individual time horizons T.

VIX squared

- Consider the payoff of a forward-starting variance swap

$$\begin{aligned}\zeta_T(T) &= \frac{1}{\Delta}\int_T^{T+\Delta} \xi_T(u)\,du \\ &= \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}_T \int_T^{T+\Delta} v_u\,du \\ &= \frac{1}{\Delta}\int_T^{T+\Delta} \mathbb{E}_T \langle X\rangle_{T,T+\Delta},\end{aligned}$$

  which, when Δ is 30 days, is just VIX squared.
- The $\mathbb{G}$-expansion gives us the joint MGF of VIX^2, X and $\langle X\rangle$ as follows.

Triple joint MGF

Theorem 3 (Theorem 4.4 of [FGR20])

For $a, b, c \in \mathbb{R}$ sufficiently small,

$$\mathbb{E}_t\left[e^{a\,X_T + b\,\langle X\rangle_{t,T} + c\,\zeta_T(T)}\right] = \exp\left\{a\,X_t + c\,\zeta_t(T) + \sum_{k=2}^{\infty}\mathbb{G}_t^k\right\},$$

where

$$\begin{aligned}\mathbb{G}^2 &= \left(\tfrac{1}{2}a\,(a-1)+b\right)(X\diamond X)_t(T) + ac\,X\diamond\zeta + \tfrac{1}{2}c^2\,\zeta\diamond\zeta,\\ \mathbb{G}^k &= \frac{1}{2}\sum_{j=2}^{k-2}\mathbb{G}^{k-j}\diamond\mathbb{G}^j + (a\,X\diamond\mathbb{G}^{k-1})\ \textit{for}\ k>2.\end{aligned}$$

Proof.

This is a direct consequence of Theorem 1: The time-T quantity of interest is

$$A_T := a\,X_T + b\,\langle X\rangle_{t,T} + c\,\zeta_T(T)$$

and it suffices to compute (using that $X + \frac{1}{2}\langle X\rangle$ is martingale),

$$\mathbb{E}_t\left[A_T\right] = a\,X_t + (b - \tfrac{1}{2}\,a)\,(X \diamond X)_t(T) + c\,\zeta_t(T)\,.$$

□

- Theorem 3 is completely *model-independent*!
  - It is useful in particular when the diamond trees are easy to compute or approximate.
- We can get the joint MGF of any set of random variables of interest in the same way.
  - For example, VIX futures are martingales. So the joint MGF of SPX and VIX is in principle computable!

Trees with colored leaves

Denote $X \equiv \bullet$ and $\zeta \equiv \bullet$.

- In Theorem 3 we wrote

$$\mathbb{G}^2 = \left(\tfrac{1}{2}a\,(a-1) + b\right) \; + ac \; + \tfrac{1}{2}c^2 \;.$$

- We could define $(X \diamond X) = M$, or $= \bullet$, resulting in trees with leaves of three different colors.
  - In a forward variance model, X_t represents the log-stock price and $M_t(T)$, the expected total variance $\int_t^T \xi_t(u)\,du$.
- Then

$$\mathbb{G}^2 = \left(\tfrac{1}{2}a\,(a-1) + b\right) \bullet + ac \; + \tfrac{1}{2}c^2 \;.$$

- In general, we can always identify subtrees in this way and assign them a new variable name (and leaf color).

$\mathbb{F}$-recursion

Putting $b = -\frac{1}{2}a$ in the $\mathbb{G}$-recursion gives the $\mathbb{F}$-recursion.

Theorem 4

With $\mathbb{F}^2 = \frac{1}{2}a\,(a-1) = \frac{1}{2}a\,(a-1)\bullet$ *and* $\forall k > 2$,

$$\mathbb{F}^k = \frac{1}{2}\sum_{j=2}^{k-2} \mathbb{F}^{k-j} \diamond \mathbb{F}^j + (a\,Y \diamond \mathbb{F}^{k-1}), \tag{5}$$

and we have, for sufficiently small a,

$$\log \mathbb{E}_t\left[e^{aX_T}\right] = a\,X_t + \sum_{k \geq 2} \mathbb{F}^k\,. \tag{6}$$

On the other hand, Corollary 3.1 of [AGR2020] reads:

Corollary

The cumulant generating function (CGF) is given by

$$\psi_t(T;a) = \log \mathbb{E}_t\left[e^{i\,a\,X_T}\right] = i\,a\,X_t - \frac{1}{2}a\,(a+i)\,M_t(T) + \sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}_\ell(a). \tag{7}$$

where the $\tilde{\mathbb{F}}_\ell$ satisfy the recursion

$\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+\mathrm{i})\,M_t = -\frac{1}{2}a(a+\mathrm{i}) \bullet$ and for $k>0$,

$$\tilde{\mathbb{F}}_\ell = \frac{1}{2}\sum_{j=0}^{\ell-2}\left(\tilde{\mathbb{F}}_{\ell-2-j}\diamond\tilde{\mathbb{F}}_j\right) + \mathrm{i}a\left(X\diamond\tilde{\mathbb{F}}_{\ell-1}\right). \tag{8}$$

- With the identification $\tilde{\mathbb{F}}_\ell = \mathbb{F}^{\ell+2}$, formulae (6) and (7), and the recursions (5) and (8) are equivalent.

Applying the recursion (8), the first few $\tilde{\mathbb{F}}$ forests are given by

$$
\begin{aligned}
\tilde{\mathbb{F}}_0 &= -\tfrac{1}{2} a(a+\mathrm{i}) \bullet \\
\tilde{\mathbb{F}}_1 &= -\tfrac{\mathrm{i}}{2} a^2 (a+\mathrm{i}) \\
\tilde{\mathbb{F}}_2 &= \tfrac{1}{2^3} a^2 \,(a+\mathrm{i})^2 + \tfrac{1}{2} a^3 \,(a+\mathrm{i}) \\
\tilde{\mathbb{F}}_3 &= (\tilde{\mathbb{F}}_0 \diamond \tilde{\mathbb{F}}_1) + \mathrm{i} a \bullet \diamond \tilde{\mathbb{F}}_2 \\
&= \tfrac{\mathrm{i}}{2^2} a^3 \,(a+\mathrm{i})^2 + \tfrac{\mathrm{i}}{2^3} a^3 \,(a+\mathrm{i})^2 + \tfrac{\mathrm{i}}{2} a^4 \,(a+\mathrm{i}) .
\end{aligned}
$$

- Note that the total probability and martingale constraints are satisfied for each tree.
  - That is $\psi_t^T(0) = \psi_t^T(-\mathrm{i}) = 0$.

Variance and gamma swaps

The variance swap is given by the fair value of the log-strip:

$$\mathbb{E}_t\left[X_T\right] = (-\mathrm{i})\,\psi_t^{T\prime}(0) = X_t - \tfrac{1}{2}\,M_t(T)$$

and the gamma swap (wlog set $X_t = 0$) by

$$\mathbb{E}_t\left[X_T\,e^{X_T}\right] = -\mathrm{i}\,\psi_t^{T\prime}(-\mathrm{i}).$$

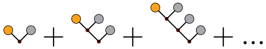
Remark

We can in principle compute such moments for any stochastic volatility model written in forward variance form, whether or not there exists a closed-form expression for the characteristic function.

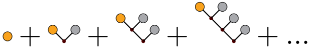
The gamma swap

It is easy to see that only trees containing a single • leaf will survive in the sum after differentiation when $a = -\mathrm{i}$ so that

$$\sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}'_\ell(-\mathrm{i}) = \frac{\mathrm{i}}{2} \sum_{\ell=1}^{\infty} X^{\diamond \ell} M$$

$= \frac{\mathrm{i}}{2} \Big\{$  $\Big\}$

Then the fair value of a gamma swap is given by

$\mathcal{G}_t(T) = 2\,\mathbb{E}_t\left[X_T\, e^{X_T}\right] =$  (9)

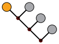
Remark

Equation (9) allows for explicit computation of the gamma swap for any model written in forward variance form.

The leverage swap

We deduce that the fair value of a leverage swap is given by

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - M_t(T) = \sum_{\ell=1}^{\infty} X^{\diamond \ell} M$$

$=$  $+$  $+$  $+ \ldots$ (10)

- The leverage swap is expressed explicitly in terms of covariance products of the spot and vol. processes.
  - If spot and vol. processes are uncorrelated, the fair value of the leverage swap is zero.

An explicit model-free expression for the leverage swap!

$\mathcal{L}_t(T)$ directly from the smile

- Let

$$d_\pm(k) = \frac{-k}{\sigma_{\rm BS}(k,T)\sqrt{T}} \pm \frac{\sigma_{\rm BS}(k,T)\sqrt{T}}{2}$$

  and following Fukasawa [Fuk12], denote the inverse functions by $g_\pm(z) = d_\pm^{-1}(z)$. Further define

$$\sigma_\pm(z) = \sigma_{\rm BS}(g_\pm(z),T)\,\sqrt{T}.$$

- It is a well-known corollary of Matytsin's characteristic function representation in [Mat00], that

$$M_t(T) = \int_{\mathbb{R}} dz\, N'(z)\,\sigma_-^2(z).$$

- The gamma swap is given by

$$\mathcal{G}_t(T) = \int_{\mathbb{R}} dz\, N'(z)\,\sigma_+^2(z).$$

Fast calibration

- For each T, $\mathcal{L}_t(T) = \mathcal{G}_t(T) - M_t(T)$ may be estimated from the observed smile.
  - In the case of SPX, there are currently between 30 and 40 listed expirations.
- Also, $\mathcal{L}_t(T) = \sum_{\ell=1}^{\infty} X^{\diamond \ell} M$.
- For models (such as affine forward variance models) where diamond trees are easily computable, fast calibration is then possible.

The Bergomi-Guyon smile expansion

- The Bergomi-Guyon (BG) smile expansion (Equation (14) of [BG12]) reads

$$\sigma_{\mathrm{BS}}(k, T) = \hat{\sigma}_T + \mathcal{S}_T \, k + \mathcal{C}_T \, k^2 + \mathcal{O}(\epsilon^3)$$

  where the coefficients $\hat{\sigma}_T$, $\mathcal{S}_T$ and $\mathcal{C}_T$ are complicated combinations of trees such as .
- The beauty of the BG expansion is that in some sense, it yields direct relationships between the smile and autocovariance functionals.

A formal expansion

Regarding the forest expansion (7) as a formal power series in ϵ whose power counts the forest index ℓ, the characteristic function of the log stock price may be written in the form

$$\varphi_t(T;a) = \exp\left\{\mathrm{i}\, a\, X_t - \frac{1}{2} a\,(a+\mathrm{i})\, M_t(T) + \sum_{\ell=1}^{\infty} \epsilon^{\ell}\, \tilde{\mathbb{F}}_{\ell}(a)\right\}.$$

On the other hand, from for example equation (5.7) of [Gat06], with $X_t = 0$,

$$\int_0^{\infty} \frac{du}{u^2+\frac{1}{4}} \mathrm{Re}\left[e^{-\mathrm{i}uk}\left(\varphi_t^T\,(u-\mathrm{i}/2) - e^{-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\Sigma(k)}\right)\right] = 0 \tag{11}$$

where $\Sigma(k) = \sigma^2_{\mathrm{BS}}(k,T)\,T$ is the implied total variance smile, $k = \log K/S$ is the log-strike, and T is time to expiration.

Let

$$\Sigma(k) = \sum_{\ell=0}^{\infty} \epsilon^\ell \, a_\ell(k).$$

Equation (11) may then be rewritten in the form

$$\begin{aligned} & \int_0^\infty \frac{du}{u^2+\frac{1}{4}} \, \mathrm{Re}\left[e^{-\mathrm{i}uk} \exp\left\{-\frac{1}{2}\left(u^2+\frac{1}{4}\right)\sum_{\ell=0}^{\infty} \epsilon^\ell \, a_\ell(k)\right\}\right] \\ = \; & \int_0^\infty \frac{du}{u^2+\frac{1}{4}} \, \mathrm{Re}\left[e^{-\mathrm{i}uk} \, e^{-\frac{1}{2}(u^2+1/4)\, M_t(T)} \exp\left\{\sum_{\ell=1}^{\infty} \epsilon^\ell \, \tilde{\mathbb{F}}_\ell(u-\mathrm{i}/2)\right\}\right]. \end{aligned} \tag{12}$$

Matching powers of ϵ on each side of (12) gives the coefficients $a_\ell(k)$ in terms of diamond trees, for any $\ell \in \mathbb{Z}^+$.

$$
\begin{aligned}
a_0(k) &= M_t(T) = \bullet \\
a_1(k) &= \left(\frac{k}{M} + \frac{1}{2}\right) \\
a_2(k) &= \frac{1}{4}(\ )^2 \left\{-\frac{5k^2}{M^3} - \frac{2k}{M^2} + \frac{3}{M^2} + \frac{1}{4M}\right\} \\
&\quad + \frac{1}{4}(\ ) \left\{\frac{k^2}{M^2} - \frac{1}{M} - \frac{1}{4}\right\} \\
&\quad + (\ ) \left\{\frac{k^2}{M^2} + \frac{k}{M} - \frac{1}{M} + \frac{1}{4}\right\}.
\end{aligned}
$$

It is straightforward to verify that the resulting expansion coincides with that of Bergomi and Guyon up to second order in ϵ.

Bergomi-Guyon to higher order

This algorithm can be extended to any desired order. For example,

$$
\begin{aligned}
a_3(k) \;=\;& \mathcal{I}_{0,3} + \left(\; + \tfrac{1}{2} \; \right) \mathcal{I}_{1,1} \\
&+ \tfrac{1}{2} \; \left[\mathcal{I}_{2,1} - 2\,\mathcal{I}_{1,0}^2\,\mathcal{I}_{0,1} \right] \\
&+ \; \left[\mathcal{I}_{1,3} - \mathcal{I}_{1,0}\,\mathcal{I}_{0,1}\,\mathcal{I}_{0,2} \right] \\
&+ \tfrac{1}{6} \left(\; \right)^3 \left[\mathcal{I}_{2,3} - \mathcal{I}_{2,0}\,\mathcal{I}_{0,1}^3 - 3\,\mathcal{I}_{1,0}\,\mathcal{I}_{0,1} \left(\mathcal{I}_{1,2} - \mathcal{I}_{1,0}\,\mathcal{I}_{0,1}^2 \right) \right].
\end{aligned}
\tag{13}
$$

- The $\mathcal{I}_{i,j}$ are Hermite-like polynomials in k.

We may compute the coefficients in (13) explicitly as follows.

$$
\begin{aligned}
\mathcal{I}_{0,3} &= \frac{k^3}{M^3} + \frac{3k^2}{2M^2} - \frac{3k}{M^2} + \frac{3k}{4M} - \frac{3}{2M} + \frac{1}{8} \\
\mathcal{I}_{1,1} &= \frac{k^3}{2M^3} + \frac{k^2}{4M^2} - \frac{3k}{2M^2} - \frac{k}{8M} - \frac{1}{4M} - \frac{1}{16} \\
\mathcal{I}_{2,1} - \mathcal{I}_{1,0}^2\,\mathcal{I}_{0,1} &= -\frac{2k^3}{M^4} - \frac{k^2}{2M^3} + \frac{k}{4M^2} + \frac{7k}{2M^3} + \frac{1}{4M^2} \\
\mathcal{I}_{1,3} - \mathcal{I}_{1,0}\,\mathcal{I}_{0,1}\,\mathcal{I}_{0,2} &= -\frac{4k^3}{M^4} - \frac{7k^2}{2M^3} - \frac{k}{2M^2} + \frac{7k}{M^3} + \frac{2}{M^2} + \frac{1}{8M} \\
\mathcal{I}_{2,3} - \mathcal{I}_{2,0}\,\mathcal{I}_{0,1}^3 &- 3\,\mathcal{I}_{1,0}\,\mathcal{I}_{0,1}\,\left(\mathcal{I}_{1,2} - \mathcal{I}_{1,0}\,\mathcal{I}_{0,1}^2\right) \\
&= \frac{39k^3}{2M^5} + \frac{45k^2}{4M^4} + \frac{3k}{8M^3} - \frac{24k}{M^4} - \frac{3}{16M^2} - \frac{9}{2M^3}.
\end{aligned}
$$

Third order skew

The ATM total variance skew is given by

$$
\begin{aligned}
\Sigma'(0) &= \sum_{\ell=0}^{3} \epsilon^{\ell}\, a'_{\ell}(0) + \mathcal{O}(\epsilon^4) \\
&= \frac{\epsilon}{M} \; + \frac{\epsilon^2}{M} \; - \frac{\epsilon^2}{2\,M^2} \left(\; \right)^2 \\
&\quad + \epsilon^3 \left(\frac{3}{4M} - \frac{3}{M^2} \right) \; + \epsilon^3 \left(-\frac{3}{2M^2} - \frac{1}{8M} \right) \left(\; + \frac{1}{2} \; \right) \\
&\quad + \epsilon^3\, \frac{1}{2} \; \left[\frac{1}{4M^2} + \frac{7}{2M^3} \right] + \epsilon^3 \; \left[-\frac{1}{2M^2} + \frac{7}{M^3} \right] \\
&\quad + \epsilon^3 \left(\; \right)^3 \left[\frac{1}{16M^3} - \frac{4}{M^4} \right] + \mathcal{O}(\epsilon^4).
\end{aligned}
$$

- Compare with the approximation

$$\Sigma'(0) \approx \frac{1}{M} \left\{ \; + \; + \; + \ldots \right\}$$

in [Fuk14].

Affine forward variance models

Following [GKR19] consider *affine forward variance models* of the form

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\
d\xi_t(u) &= \kappa(u-t)\,\sqrt{v_t}\, dW_t,
\end{aligned}
$$

with $d\langle W, Z\rangle_t = \rho\, dt$.

- This class of models includes classical and rough Heston.
- As we will see, diamond trees are particularly easy to compute in AFV models.

Affine trees

Lemma 5 (Lemma 4.5 of [FGR20])

In an affine forward variance model, all diamond trees take the form

$$\int_t^T \xi_t(u)\, h(T-u)\, du$$

for some function h.

Classical Heston

Example (Classical Heston)

In this case,

$$d\xi_t(u) = \nu\, e^{-\lambda\,(u-t)}\, \sqrt{v_t}\, dW_t.$$

Then, for example,

$$= (X \diamond M)_t(T) \quad = \quad \frac{\rho\,\nu}{\lambda} \int_t^T \xi_t(u)\, \left[1 - e^{-\lambda(T-u)}\right]\, du.$$

Rough Heston

Example (Rough Heston)

In this case, with $\alpha = H + 1/2 \in (1/2, 1)$ *(and with* $\lambda = 0$*),*

$$d\xi_t(u) = \frac{\nu}{\Gamma(\alpha)}\,(u-t)^{\alpha-1}\,\sqrt{v_t}\,dW_t.$$

Then, for example,

$$\begin{aligned}
\bullet = M_t(T) &= (X \diamond X)_t(T) = \int_t^T \xi_t(u)\,du, \\
\vee &= \frac{\nu^2}{\Gamma(\alpha)^2}\int_t^T \xi_t(u)\,du\left(\int_u^T (s-u)^{\alpha-1}\,ds\right)^2 \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2}\int_t^T \xi_t(u)\,(T-u)^{2\alpha}\,du.
\end{aligned}$$

- For a bounded forward variance curve ξ one then sees that diamond trees with k leaves are of order $(T-t)^{1+(k-2)\alpha}$.
- In this case, the $\mathbb{F}$-expansion (forest reordering according to number of leaves) has the interpretation of a short-time expansion, the concrete powers of which depend on the roughness parameter $\alpha = H + 1/2 \in (1/2, 1)$, cf. [CGP21, GR19].

The triple joint MGF in affine forward variance models

- Lemma 5 combined with Theorem 3 characterize the triple-joint MGF of X_T, $\langle X \rangle_T$ and $\zeta_T(T)$ for an affine forward variance model.
  - Compare with Theorem 4.3 of [AJLP2019] and Proposition 4.6 of [GKR19].
- We obtain the convolutional form

$$\mathbb{E}_t \left[e^{a\, X_T + b\, \langle X \rangle_{t,T} + c\, \zeta_T(T)} \right] = \exp\left\{ a\, X_t + (\xi \star g)(\tau; a, b, c)_t(T) \right\} \ .$$

- This is consistent with (and generalizes) Theorem 2.6 of [GKR19] where the same convolution Riccati equation appears, but with $g = g(\tau; a)$ instead of $(\tau; a, b, c)$ and different boundary conditions.

Computation of trees under rough Heston

Abbreviating bounded variation terms as 'BV', we have

$$
\begin{aligned}
dX_t &= \sqrt{v_t}\,dZ_t + \mathrm{BV} \\
dM_t &= \int_t^T d\xi_t(u)\,du + \mathrm{BV} \\
&= \frac{\nu}{\Gamma(\alpha)}\sqrt{v_t}\left(\int_t^T \frac{du}{(u-t)^\gamma}\right) dW_t + \mathrm{BV} \\
&= \frac{\nu\,(T-t)^\alpha}{\Gamma(1+\alpha)}\sqrt{v_t}\,dW_t + \mathrm{BV}\,.
\end{aligned}
$$

The first order forest

There is only one tree in the forest $\tilde{\mathbb{F}}_1$.

$\tilde{\mathbb{F}}_1 =$  $= (X \diamond M)_t(T)$

$$
\begin{aligned}
(X \diamond M)_t(T) &= \mathbb{E}_t\left[\int_t^T d\langle X, M\rangle_s\right] \\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\,\mathbb{E}_t\left[\int_t^T v_s\,(T-s)^\alpha\,ds\right] \\
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\int_t^T \xi_t(s)\,(T-s)^\alpha\,ds.
\end{aligned}
$$

Higher order forests

Define for $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds\, \xi_t(s)\, (T-s)^{j\,\alpha}.$$

Then

$$\begin{aligned} dI_s^{(j)}(T) &= \int_s^T du\, d\xi_s(u)\, (T-u)^{j\,\alpha} + \mathrm{BV} \\ &= \frac{\nu\,\sqrt{v_s}}{\Gamma(\alpha)}\, dW_s \int_s^T \frac{(T-u)^{j\,\alpha}}{(u-s)^{\gamma}}\, du + \mathrm{BV} \\ &= \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)}\, \nu\,\sqrt{v_s}\, (T-s)^{(j+1)\,\alpha}\, dW_s + \mathrm{BV}. \end{aligned}$$

With this notation,

 $$= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, I_t^{(1)}(T).$$

The second order forest

There are two trees in $\tilde{\mathbb{F}}_2$:

$$
\begin{aligned}
&= \mathbb{E}_t\left[\int_t^T d\langle M, M\rangle_s\right] \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2}\int_t^T \xi_t(s)\,(T-s)^{2\,\alpha}\,ds \\
&= \frac{\nu^2}{\Gamma(1+\alpha)^2}\, I_t^{(2)}(T)
\end{aligned}
$$

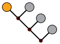
and

$$
\begin{aligned}
&= \frac{\rho\,\nu}{\Gamma(1+\alpha)}\,\mathbb{E}_t\left[\int_t^T d\langle X, I^{(1)}\rangle_s\right] \\
&= \frac{\rho^2\,\nu^2}{\Gamma(1+2\,\alpha)}\, I_t^{(2)}(T).
\end{aligned}
$$

The third order forest

Continuing to the forest $\tilde{\mathbb{F}}_3$, we have the following.

$$= \frac{\rho\,\nu^3}{\Gamma(1+\alpha)\,\Gamma(1+2\,\alpha)}\,I_t^{(3)}(T)$$



$$= \frac{\rho^3\,\nu^3}{\Gamma(1+3\,\alpha)}\,I_t^{(3)}(T)$$

$$= \frac{\rho\,\nu^3\,\Gamma(1+2\,\alpha)}{\Gamma(1+\alpha)^2\,\Gamma(1+3\,\alpha)}\,I_t^{(3)}(T).$$

In particular, we readily identify the pattern

$$\left(X^{\diamond \ell} M\right)_t(T) = \frac{(\rho\,\nu)^\ell}{\Gamma(1+\ell\,\alpha)}\,I_t^{(\ell)}(T).$$

The leverage swap under rough Heston

Using (10), we have

$$\begin{aligned}\mathcal{L}_t(T) &= \sum_{\ell=1}^{\infty} \left(X^{\diamond \ell} M\right)_t (T) \\ &= \sum_{\ell=1}^{\infty} \frac{(\rho\,\nu)^\ell}{\Gamma(1+\ell\,\alpha)} \int_t^T du\, \xi_t(u)\,(T-u)^{\ell\,\alpha} \\ &= \int_t^T du\, \xi_t(u)\, \left\{E_\alpha(\rho\,\nu\,(T-u)^\alpha) - 1\right\}\end{aligned}$$

where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function.

An explicit expression for the leverage swap!

- Since we can impute the leverage swap $\mathcal{L}_t(t)$ from the smile for each expiration T, fast calibration of the rough Heston model is possible.

Summary

- We introduced the diamond product.
- We defined the $\mathbb{G}$-expansion and gave an idea of its proof.
  - The cumulant expansion of [LRV19] and the Exponentiation Theorem of [AGR2020] are special cases.
- We showed how easy computations can be in affine forward variance models.
  - Quick calibration of such models is one application.

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